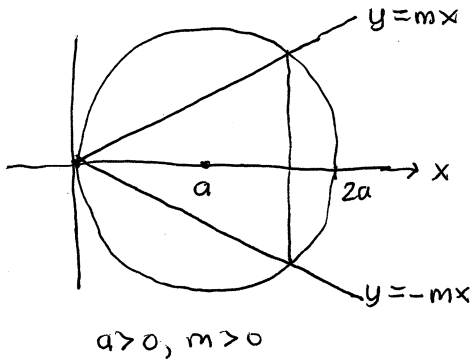


Using inverse trig functions in polar coordinate integration

an example

$$(x-a)^2 + y^2 = a^2$$



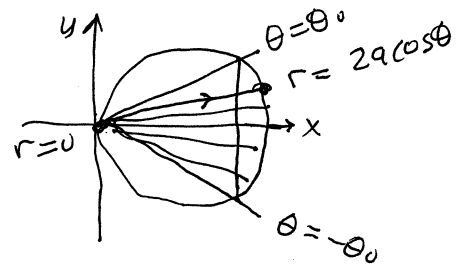
Find the centroid of the region of the plane inside the circle $(x-a)^2 + y^2 = a^2$, $a > 0$ and in between the two lines $y = \pm mx$, $m > 0$.

solution: $(x-a)^2 + y^2 = a^2$
 $x^2 - 2ax + a^2 + y^2 = a^2$
 $x^2 + y^2 = 2ax$
 $r^2 = 2a r \cos \theta$
 $r = 2a \cos \theta$

$y = mx \rightarrow \theta = \arctan \frac{y}{x} = \arctan m = \theta_0 > 0$

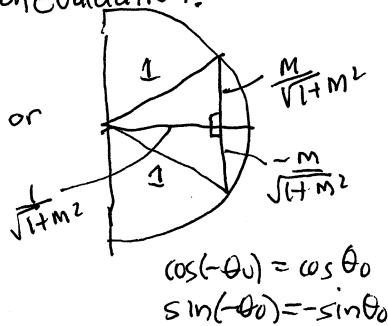
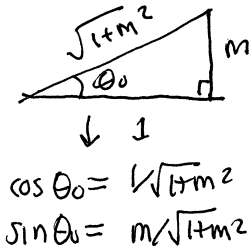
$y = -mx: \theta = -\theta_0$

integration scheme:



$\theta = -\theta_0 \dots \theta_0, r = 0 \dots 2a \cos \theta$

inverse trig function evaluation:



this is what we need to be able to evaluate all the trig functions which appear in the angular integral antiderivative

$$\langle A, A_y, A_x \rangle = \iint_R \langle 1, x, y \rangle dA = \int_{-\theta_0}^{\theta_0} \int_0^{2a \cos \theta} \langle 1, r \cos \theta, r \sin \theta \rangle r dr d\theta$$

$$= \int_{-\theta_0}^{\theta_0} \left\langle \int_0^{2a \cos \theta} r dr, \int_0^{2a \cos \theta} r^2 dr \cos \theta, \int_0^{2a \cos \theta} r^2 dr \sin \theta \right\rangle d\theta$$

$$= \left\langle \int_{-\theta_0}^{\theta_0} \frac{4a^2}{2} \cos^2 \theta d\theta, \int_{-\theta_0}^{\theta_0} \frac{8a^3}{3} \cos^3 \theta d\theta, \int_{-\theta_0}^{\theta_0} \frac{8a^3}{3} \cos^3 \theta \sin \theta d\theta \right\rangle$$

$$= \left\langle 2a^2 \left(\frac{1}{2} \cos \theta \sin \theta + \frac{1}{2} \theta \right) \Big|_{-\theta_0}^{\theta_0}, \frac{8a^3}{3} \frac{1}{4} \left[\cos^3 t + \frac{3}{2} \cos t \right] \sin t + \frac{3}{2} t \Big|_{-\theta_0}^{\theta_0}, \frac{8a^3}{3} \left(-\frac{\cos^4 \theta}{4} \right) \Big|_{-\theta_0}^{\theta_0} \right\rangle$$

$$= \left\langle 2a^2 (\cos \theta_0 \sin \theta_0 + \theta_0), \frac{4a^3}{3} (\cos^3 \theta_0 + \frac{3}{2} \cos \theta_0) \sin \theta_0 + 3\theta_0, 0 \right\rangle$$

$$= \left\langle 2a^2 \left[\left(\frac{m}{1+m^2} \right) + \arctan m \right], \frac{4a^3}{3} \left(\frac{1}{1+m^2} + \frac{3}{2} \right) \left(\frac{m}{1+m^2} \right) + 3 \arctan m, 0 \right\rangle$$

$\bar{x} = \frac{A_y}{A} = \dots$

Maple: $\lim_{m \rightarrow \infty} \bar{x} = a, \lim_{m \rightarrow 0} \bar{x} = \frac{4}{3}a = \frac{2}{3}(2a)$

isocles triangle bisectors meet at $\frac{2}{3}$ bisector