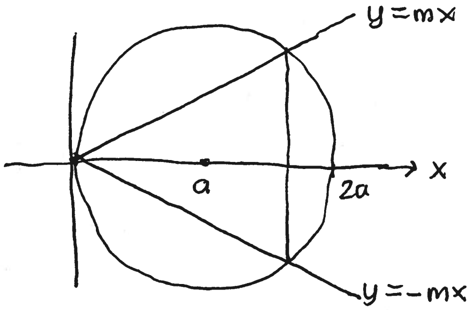
using inverse trig functions in polar coordinate integration

an example



$a>0,\ m>0$

Find the centroid of the region of the plane inside the circle $(x-a)^2+y^2=a^2$, $a>0$ and in between the two lines $y=\pm mx$, $m>0$.

solution: $(x-a)^2+y^2=a^2$

$x^2-2ax+a^2+y^2=a^2$

$x^2+y^2=2ax$

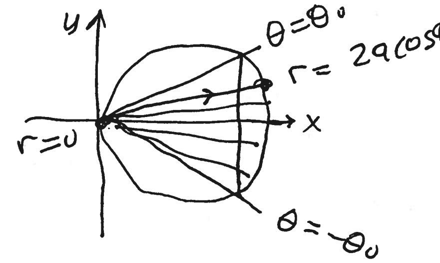
$r^2=2ar\cos\theta$

$r=2a\cos\theta$

$y=mx \to \theta=\arctan\frac{y}{x}=\arctan m=\theta_0>0$

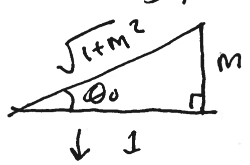
$\hookrightarrow y=-mx:\ \theta=-\theta_0$

integration scheme:



$\theta=-\theta_0..\theta_0,\ r=0..2a\cos\theta$

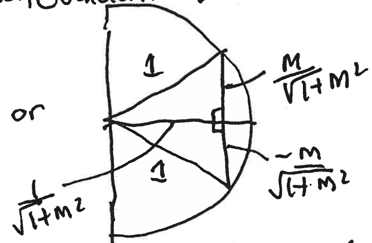
inverse trig function evaluation:



$\cos\theta_0=1/\sqrt{1+m^2}$

$\sin\theta_0=m/\sqrt{1+m^2}$

or



$\cos(-\theta_0)=\cos\theta_0$

$\sin(-\theta_0)=-\sin\theta_0$

this is what we need to be able to evaluate all the trig functions which appear in the angular integral antiderivative

$$\langle A, A_y, A_x\rangle=\iint_R \langle 1,x,y\rangle\, dA=\int_{-\theta_0}^{\theta_0}\int_0^{2a\cos\theta}\langle 1, r\cos\theta, r\sin\theta\rangle\, r\,dr\,d\theta$$

$$=\int_{-\theta_0}^{\theta_0}\Big\langle \underbrace{\int_0^{2a\cos\theta} r\,dr}_{\frac{r^2}{2}\Big|_0^{2a\cos\theta}},\ \underbrace{\int_0^{2a\cos\theta} r^2\,dr}_{\frac{r^3}{3}\Big|_0^{2a\cos\theta}}\cos\theta,\ \underbrace{\int_0^{2a\cos\theta} r^2\,dr}\sin\theta\Big\rangle\, d\theta$$

$$=\Big\langle \int_{-\theta_0}^{\theta_0}\frac{4a^2}{2}\cos^2\theta\, d\theta,\ \int_{-\theta_0}^{\theta_0}\frac{8}{3}a^3\cos^4\theta\, d\theta,\ \int_{-\theta_0}^{\theta_0}\frac{8}{3}a^3\cos^3\theta\sin\theta\, d\theta\Big\rangle$$

$$=\Big\langle 2a^2\big(\overbrace{\tfrac{1}{2}\cos\theta\sin\theta}^{\text{odd in }\theta}+\tfrac{1}{2}\theta\big)\Big|_{-\theta_0}^{\theta_0},\ \frac{8a^3}{3}\frac{1}{4}\Big[\overbrace{(\cos^3 t+\tfrac{3}{2}\cos t)\sin t}^{\text{odd in }\theta}+\tfrac{3}{2}t\Big]_{-\theta_0}^{\theta_0},\ \frac{8a^3}{3}\Big(-\frac{\cos^4\theta}{4}\Big)\Big|_{-\theta_0}^{\theta_0}\Big\rangle$$

$$=\Big\langle 2a^2(\cos\theta_0\sin\theta_0+\theta_0),\ \frac{4a^3}{3}(\cos^3\theta_0+\tfrac{3}{2}\cos\theta_0)\sin\theta_0+3\theta_0,\ 0\Big\rangle$$

$$=\Big\langle 2a^2\Big[\Big(\frac{m}{1+m^2}\Big)+\arctan m\Big],\ \frac{4}{3}a^3\Big(\frac{1}{1+m^2}+\frac{3}{2}\Big)\Big(\frac{m}{1+m^2}\Big)+3\arctan m,\ 0\Big\rangle$$

$\bar{x}=\frac{A_y}{A}=\dots$

Maple: $\lim_{m\to\infty}\bar{x}=a$, $\lim_{m\to 0}\bar{x}=\frac{4}{3}a=\frac{2}{3}(2a)$

isoceles triangle bisectors meet at 2/3 bisector