integration over circular disk passing through origin, center on axis

$\iint_R x^2+y^2\,dA$ — $\underbrace{x^2+y^2}_{\text{rotational symmetry}}$

R: circle of radius 2 with center at $(2,0)$ in xy-plane.
(rotational symmetry about different center)

we could adapt our polar coordinates to the integrand or to the region of integration. for practice let's use the usual polar coords (r,θ) and also describe the region using the Cartesian coordinates.

$(x-2)^2+y^2=4 \longrightarrow x^2-4x+4+y^2=4$

$x^2+y^2=4x \xrightarrow{\text{solve for } y} y^2=4x-x^2$

$y=\pm\sqrt{4x-x^2}$

$\downarrow$ $x=r\cos\theta$

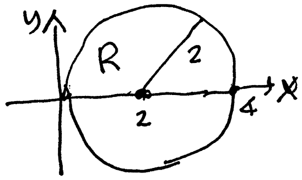
$r^2=4r\cos\theta$

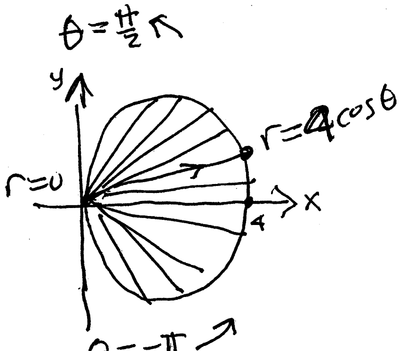
$r=4\cos\theta$

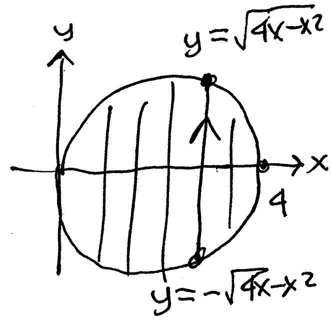
solve for x: $x^2-4x+y^2=0$

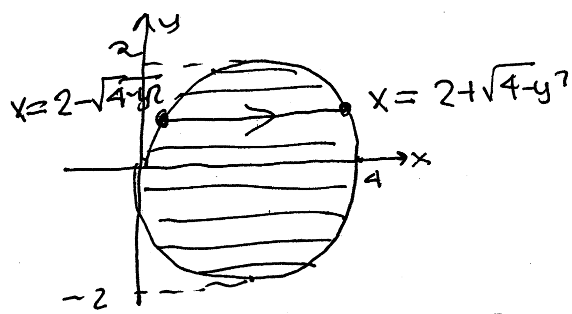
$x=\dfrac{4\pm\sqrt{16-4y^2}}{2}$

$=2\pm\sqrt{4-y^2}$









$r=0\,..\,4\cos\theta$ while $\theta=-\frac{\pi}{2}\,..\,\frac{\pi}{2}$

$y=-\sqrt{4x-x^2}\,..\,\sqrt{4x-x^2}$ while $x=0\,..\,4$

$x=2-\sqrt{4-y^2}\,..\,2+\sqrt{4-y^2}$ while $y=-2\,..\,2$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{4\cos\theta} r^2\cdot r\,dr\,d\theta=\int_0^4\int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}} x^2+y^2\,dy\,dx=\int_{-2}^{2}\int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} x^2+y^2\,dx\,dy$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_0^{4\cos\theta}\,d\theta$$

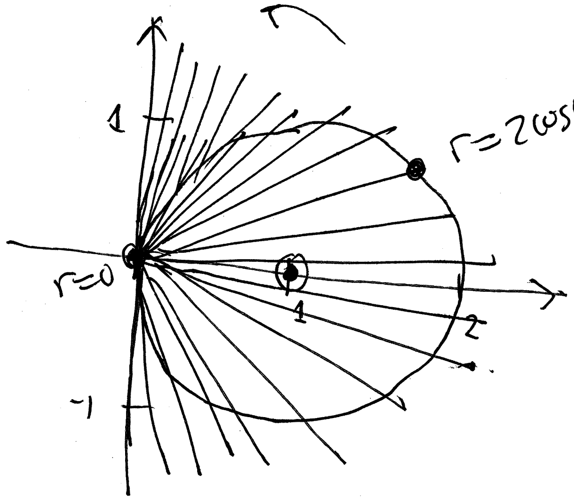
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^3\cos^4\theta\,d\theta=24\pi$$

Green's Thm 3 ways:

$\vec{F}=\langle -x^2y,\ xy^2\rangle \rightarrow \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=x^2+y^2$

so this integral is the "RHS" of Green's Thm for the region R.
Do the line integral around its boundary using all three representations of the circles / semicircles.

centroid of circular disk touching origin, center on axis



$$\boxed{\begin{array}{l}\theta = -\frac{\pi}{2} .. \frac{\pi}{2} \\ r = 0 .. 2\cos\theta\end{array}}$$

$$(x-1)^2 + y^2 = 1$$

$$x^2 - 2x + 1 + y^2 = 1$$

$$\underbrace{x^2+y^2}_{} - 2x = 0$$

$$r^2 - 2(r\cos\theta) = 0$$

$$r - 2\cos\theta = 0$$

$$r = 2\cos\theta = 0 \rightarrow \theta = \pm\frac{\pi}{2}$$

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r\,dr\,d\theta = \pi$$

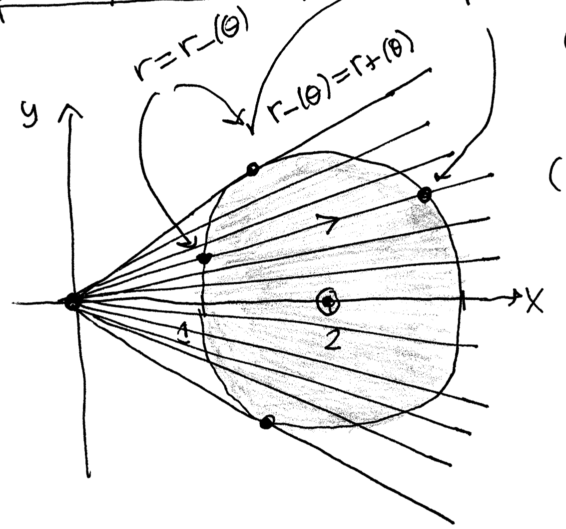
$$M_y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} \underbrace{(r\cos\theta)}_{x} r\,dr\,d\theta = \pi$$

$$M_x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} (r\sin\theta) r\,dr\,d\theta = 0$$

$$\langle \bar{x}, \bar{y} \rangle = \frac{1}{A}\langle M_y, M_x \rangle = \frac{\langle \pi, 0 \rangle}{\pi} = \langle 1, 0 \rangle.$$

center of the circle

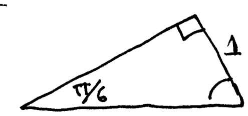
$\boxed{\text{centroid of a circular disk displaced from origin}}$



$$(x-2)^2+y^2=1$$
$$x^2-4x+4+y^2=1$$
$$(x^2+y^2)-4x+3=0$$
$$r^2-4r\cos\theta+3=0$$
$$r=\frac{4\cos\theta\pm\sqrt{16\cos^2\theta-4(3)}}{2}$$
$$=2\cos\theta\pm\sqrt{4\cos^2\theta-3}\equiv a\pm b$$
$$=r_\pm(\theta)$$

$\underline{r_+(\theta)=r_-(\theta)!}$

$$2\cos\theta+\sqrt{4\cos^2\theta-3}=2\cos\theta-\sqrt{4\cos^2\theta-3}$$
$$\hookrightarrow\ \sqrt{4\cos^2\theta-3}=0$$
$$4\cos^2\theta-3=0$$
$$\cos^2\theta=3/4$$
$$\cos\theta=\pm\frac{\sqrt3}{2}\rightarrow\frac{\sqrt3}{2}\quad \text{1st, 4th quad.}$$
$$\theta=\pm\frac{\pi}{6}\rightarrow\boxed{\begin{array}{l}\theta=-\frac{\pi}{6}\,..\,\frac{\pi}{6}\\ r=r_-(\theta)\,..\,r_+(\theta)\end{array}}$$

[

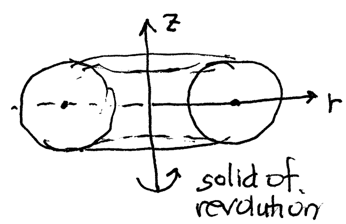
angle obvious from this simple geometry!]

$$A=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\underbrace{\int_{r_-(\theta)}^{r_+(\theta)}r\,dr}_{\underbrace{\left.\frac{r^2}{2}\right|_{r_-(\theta)}^{r_+(\theta)}}_{\frac{r_+^2(\theta)-r_-^2(\theta)}{2}}}d\theta\ \underset{\text{maple}}{=}\ \pi=\pi(1)^2!$$

$$\frac{r_+^2(\theta)-r_-^2(\theta)}{2}\leftarrow\left\{\frac{(a+b)^2-(a-b)^2}{2}=\frac{a^2+2ab+b^2-(a^2-2ab+b^2)}{2}=2ab\right.$$

$$\|$$

[in the context of cylindrical and spherical coords this allows us to integrate over a torus (donut)



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area "moments" $2(2\cos\theta)\sqrt{4\cos^2\theta-3}$

$$M_y=\iint(r\cos\theta)\,r\,dr\,d\theta=2\pi\qquad \bar{x}=\frac{M_y}{A}=\frac{2\pi}{\pi}=2\qquad (\bar{x},\bar{y})=(2,0)$$

center of disk

$$M_x=\iint(r\sin\theta)\,r\,dr\,d\theta=0\qquad \bar{y}=\frac{M_x}{A}=0$$