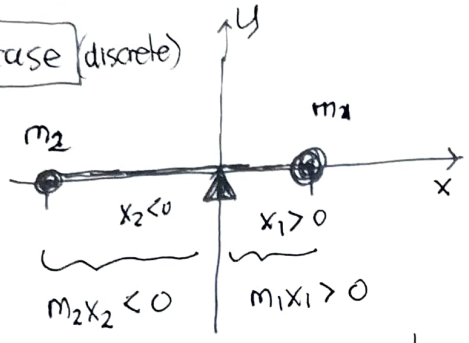
Center of Mass (Centroid - geometric center) (1)

$\boxed{\text{1d case}}$ (discrete)



imagine balancing 2 masses on the x-axis connected by a massless stiff wire, resting at the origin

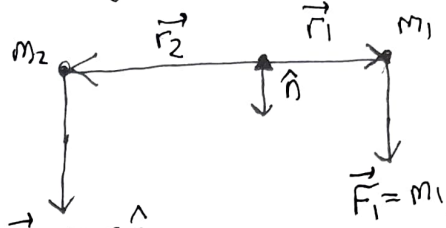
$m_2|x_2| = m_1|x_1|$ balances the moment arm magnitudes to prevent rotation

$\downarrow$

$-m_2x_2 = m_1x_1$

$m_1x_1 + m_2x_2 = 0$ balance condition $\rightarrow$ 1d center of mass is at origin

physics background (see wiki torque)



$\vec{F}_2 = m_2 g \hat{n}$ $\qquad$ $\vec{F}_1 = m_1 g \hat{n}$

$\vec{\tau}_2 = \vec{r}_2 \times \vec{F}_2$ $\odot$ out of paper clockwise torque

$\vec{\tau}_1 = \vec{r}_1 \times \vec{F}_1$ $\otimes$ into paper counter clockwise torque

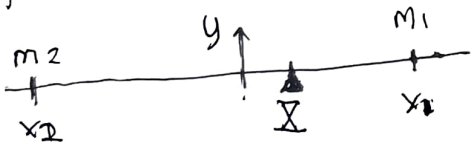
(right hand rule)

cancel if magnitudes are same

$|\vec{r}_2| m_2 g = |\vec{r}_1| m_1 g$ cancel g

$m_2|\vec{r}_2| = m_1|\vec{r}_1|$ moment arms agree

If not balanced about origin, how to find balance location?



$m_1(x_1-\bar{X}) + m_2(x_2-\bar{X}) = 0$ new balance at $x=\bar{X}$

$m_1x_1 + m_2x_2 = (m_1+m_2)\bar{X}$

$$\bar{X} = \frac{m_1x_1+m_2x_2}{m_1+m_2} = \underbrace{\left(\frac{m_1}{m_1+m_2}\right)}_{} x_1 + \underbrace{\left(\frac{m_2}{m_1+m_2}\right)}_{} x_2$$

fractional masses sum to one

$\leftarrow$ "weighted" average, each position weighted by fractional weight

if $m_1 = m_2$:

$\bar{X} = \frac{1}{2}x_1 + \frac{1}{2}x_2$ ordinary average

"geometric center"

$\downarrow$

$\bar{x}$ overbar denotes center of mass value of x

We can extend this to any number of masses:

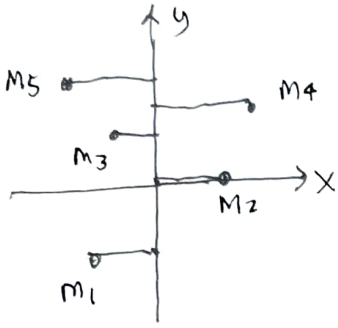
$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\underbrace{\sum_{i=1}^{n} m_i}_{\equiv M}} = \sum_{i=1}^{n} \underbrace{\left(\frac{m_i}{M}\right)}_{} x_i \xrightarrow[\text{masses}]{\text{equal}} \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{(ordinary average)}$$

(fractional mass: $\sum_{i=1}^{n} \frac{m_i}{M} = \frac{\sum_{i=1}^{n} m_i}{M} = \frac{M}{M} = 1$

for equal masses $\frac{m_i}{\sum_{i=1}^{n} m_i} = \frac{m}{nm} = \frac{1}{n}$)

Center of Mass (Centroid) (2)

2d case (discrete)

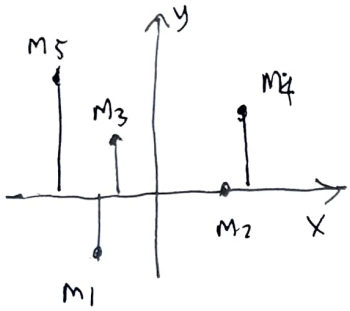


"moment about y-axis" of mass distribution

$$M_y = \sum_{i=1}^{n} m_i x_i$$ (weighted sum of signed distances from y axis)

$$M = \sum_{i=1}^{n} m_i$$ (total mass)

$$\bar{x} \equiv \frac{M_y}{M} = \sum_{i=1}^{n} \frac{m_i x_i}{M}$$ (x coord of center of mass point)

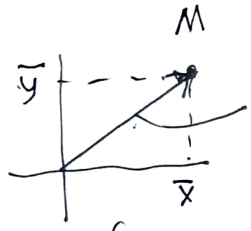


"moment about x-axis" of mass distribution

$$M_x = \sum_{i=1}^{n} m_i y_i \qquad \bar{y} = \frac{M_x}{M} = \sum_{i=1}^{n} \left(\frac{m_i}{M}\right) y_i$$

Note $M_y = M\bar{x}$, $M_x = M\bar{y}$

equivalent to putting all mass at point $(\bar{x}, \bar{y})$, will have same moments



center of mass position vector

$$\vec{r} = \langle \bar{x}, \bar{y} \rangle = \left\langle \sum_{i=1}^{n} \frac{m_i}{M} x_i, \sum_{i=1}^{n} \frac{m_i}{M} y_i \right\rangle = \underbrace{\sum_{i=1}^{n} \frac{m_i}{M} \langle x_i, y_i \rangle}_{\text{weighted average of position vector}}$$

$$= \frac{\langle M_x, M_y \rangle}{M}$$

equal mass case

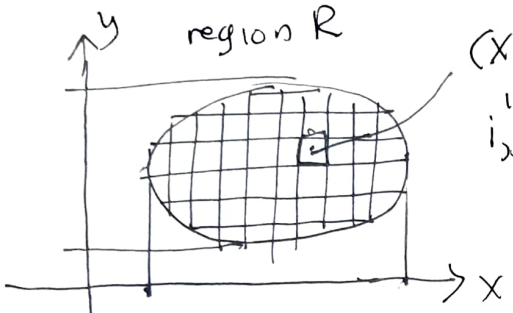
$\frac{m_i}{M} = \frac{1}{n}$ get ordinary average of position vectors

= geometric center

= "centroid" position vector

Center of Mass (Centroid) (3)

$\boxed{\text{2d case}}$ continuous distribution of mass on flat surface of zero thickness ("surface mass" distribution)



(x_i^*, y_j^*) in i,j th box

$$\underbrace{\Delta m_{i,j}}_{\text{mass in } i,j \text{ box}} = \underbrace{\rho(x_i^*, y_j^*)}_{\text{mass per unit area in box}} \underbrace{\Delta A}_{\text{area of box}}$$

grid $\begin{cases} x_i,\ i=1..n : \Delta x \\ y_j,\ j=1..m,\ \Delta y \end{cases}$

$\Delta A = \Delta x \Delta y$

total moments about y and x axes

Riemann:

$$\vec{r} = \langle \bar{x}, \bar{y} \rangle = \frac{\lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta m_{i,j} \vec{r}_{i,j}^{\,*} \Big\} = \langle M_y, M_x \rangle}{\lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta m_{i,j} \Big\} = M \text{ total mass}}$$

"homogeneous" distribution

$$= \frac{\iint_R \rho(x,y) \langle x,y \rangle \, dA}{\iint_R \rho(x,y) \, dA}$$

calculate separately, then divide

If $\rho(x,y) = \rho_0$ these cancel, get centroid

$$\langle \bar{x}, \bar{y} \rangle = \frac{\iint \langle x,y \rangle dA}{A}$$

$$= \iint \underbrace{\frac{\rho(x,y)}{M}}_{\text{fractional mass weighting factor:}} \langle x,y \rangle dA = \left\langle \frac{M_y}{M}, \frac{M_x}{M} \right\rangle = \langle \bar{x}, \bar{y} \rangle$$

$$\iint \frac{\rho(x,y)}{M} dA = \frac{\iint \rho(x,y) \, dA}{M} = \frac{M}{M} = 1$$

Step by Step:

calculate: $M_y = \iint_R \rho(x,y)\, x \, dA$, $M_x = \iint_R \rho(x,y)\, y \, dA$, $M = \iint_R \rho(x,y) \, dA$

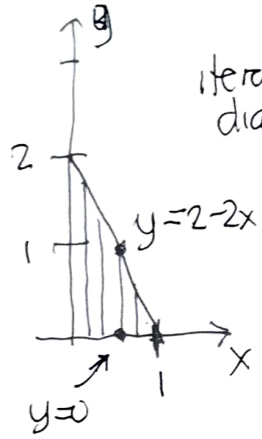
then take ratios: $\bar{x} = \frac{M_y}{M}$, $\bar{y} = \frac{M_x}{M}$

our goal here is only to evaluate these integrals for various regions and densities. (set $\rho = 1$ to get centroid)

Center of Mass (Centroid) (4)

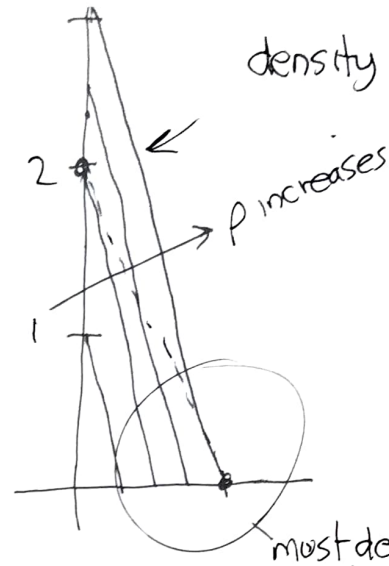
Example

triangular region



iteration diagram

R: $y = 0 \,..\, 2-2x$
while $x = 0 \,..\, 1$

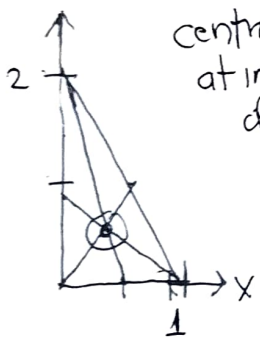


density $\rho(x,y) = 1+3x+y$
$= \rho_0$
level curves

$y = \rho_0 - 1 - 3x$
$m = -3$
steeper than
$m = -2$
of hypotenuse

most dense region is at lower right corner

expect geometric center to shift to the right and down by this unequal weighting relative to normal line to contours



centroid is at intersection of 3 bisectors

expect center of mass to shift down and to the right a bit in direction of increasing density

oops!

insert $\rho = 1+3x+y$!

$$\langle M, M_y, M_x \rangle = \left\langle \iint_R \rho\, 1\, dA,\ \iint_R \rho x\, dA,\ \iint_R \rho y\, dA \right\rangle$$

$$= \left\langle \int_0^1\!\!\int_0^{2-2x} \rho\, 1\, dy\, dx,\ \int_0^1\!\!\int_0^{2-2x} \rho x\, dy\, dx,\ \int_0^1\!\!\int_0^{2-2x} \rho y\, dy\, dx \right\rangle$$

$$= \int_0^1\!\!\int_0^{2-2x} \rho \langle 1, x, y \rangle\, dy\, dx$$

more efficient notation but equal to previous line!

Calculate these 3 integrals, then the two ratios:

$$\bar{x} = \frac{M_y}{M},\quad \bar{y} = \frac{M_x}{M} \qquad \text{or} \quad \langle \bar{x}, \bar{y} \rangle = \frac{\langle M_y, M_x \rangle}{M}$$

Find: $\langle M, M_y, M_x \rangle = \left\langle \frac{8}{3}, 1, \frac{11}{6} \right\rangle$

$$\langle \bar{x}, \bar{y} \rangle = \frac{\langle 1, 11/6 \rangle}{8/3} = \left\langle \frac{3}{8}, \frac{11}{16} \right\rangle = \underbrace{\langle 0.375, 0.688 \rangle}_{\text{need this for interpretation}}$$

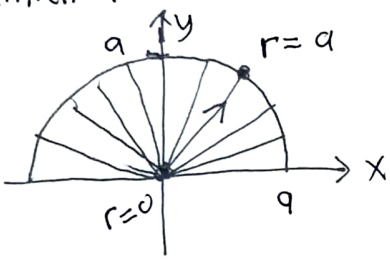
(see Maple worksheet)

compared with centroid

Center of Mass (Centroid) (5)

Example

semicircular disk



R: $r = 0..a$
while $\theta = 0..\pi$

ρ oc distance from center
$= kr$ ($k > 0$ constant)

$$\langle M, M_y, M_x \rangle = \int_0^{\pi}\int_0^{a} \langle \rho, x\rho, y\rho \rangle \underbrace{r}_{\text{correction factor}} \overbrace{dr\,d\theta}^{dA}$$

$$= \int_0^{\pi}\int_0^{a} \langle kr, kr^2\cos\theta, kr^2\sin\theta \rangle\, r\,dr\,d\theta$$

$$= k\int_0^{\pi}\int_0^{a} \langle r^2, r^3\cos\theta, r^3\sin\theta \rangle\, dr\,d\theta$$

constant limits of integration
integrands factor into separate functions of r and θ
so double integral factors into product of each variable integral

by symmetry across y-axis, must lie on y-axis!

$$= k\left\langle \underbrace{\int_0^a r^2 dr}_{\frac{a^3}{3}} \underbrace{\int_0^{\pi} 1\, d\theta}_{\pi},\ \underbrace{\int_0^a r^3 dr \underbrace{\int_0^{\pi} \cos\theta\, d\theta}_{\sin\theta\big|_0^{\pi} = 0}}_{\to \bar{x} = 0},\ \underbrace{\int_0^a r^3 dr}_{\frac{a^4}{4}} \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{-\cos\theta\big|_0^{\pi} = 2} \right\rangle$$

$$= k\left\langle \frac{\pi a^3}{3}, 0, \frac{a^4}{2} \right\rangle$$

irrelevant
(but $M = \frac{\pi a^3}{3}k \to k = \frac{3M}{\pi a^3}$)

$$\langle \bar{x}, \bar{y} \rangle = \frac{\langle 0, a^4/2 \rangle}{\pi a^3/3} = \langle 0, \underbrace{\tfrac{3}{2\pi}}_{0.477} a \rangle = a\langle 0, \tfrac{3}{2\pi} \rangle$$

just scales up the unit circle result, carries length dimension

just below the halfway mark on the y axis.

density increases upwards, shifts up from centroid

compare to centroid:

$$\langle M, M_y, M_x \rangle = \int_0^{\pi}\int_0^{a} \langle 1, r\cos\theta, r\sin\theta \rangle\, r\,dr\,d\theta$$

$$= \left\langle \underbrace{\int_0^a r\,dr \int_0^{\pi} d\theta}_{\frac{a^2}{2}(\pi)},\ \underbrace{\int_0^a r^2 dr \int_0^{\pi} \cos\theta\, d\theta}_{\frac{a^3}{3}(0)},\ \underbrace{\int_0^a r^2 dr \int_0^{\pi} \sin\theta\, d\theta}_{\frac{a^3}{3}(2)} \right\rangle$$

$$\bar{y} = \frac{2a^3/3}{\pi a^2/2} = \left(\frac{4}{3\pi}\right)a \approx 0.424\,a$$

(see Maple Worksheet)