integration over 2D and 3D regions of the plane and space

setting up iterated double or triple integrals is all about parametrizing these regions by giving a set of nested functional relationships which specifying the starting and stopping values of the coordinates

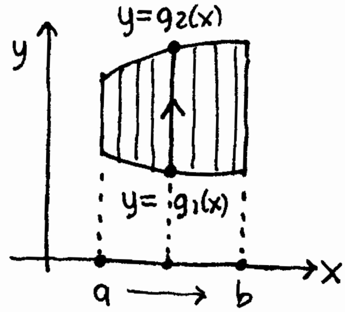
coords $\underbrace{\underbrace{\underbrace{(u}_{}, v}_{2D}, w)}_{3D}$

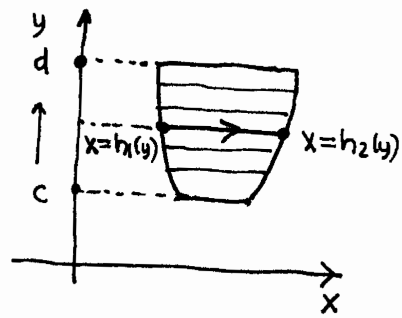
starting/stopping values:

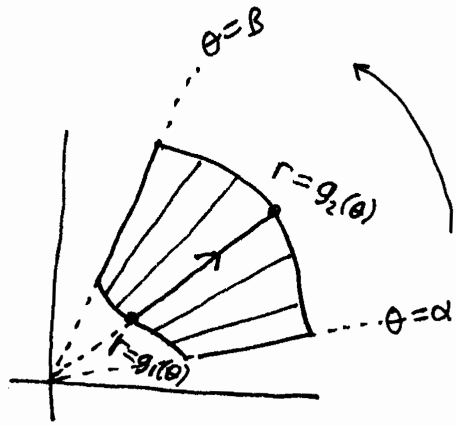
$$\left.\begin{array}{l} w = h_1(u,v) .. h_2(u,v) \\ 2D \left\{ \begin{array}{l} v = g_1(u) .. g_2(u) \\ u = a .. b \end{array} \right. \end{array} \right\} 3D$$

2D regions R:

$$\iint_R f(x,y)\, dA$$







$y = g_1(x) .. g_2(x)$
$x = a .. b$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, \underbrace{dy\, dx}_{dA}$$

$x = h_1(y) .. h_2(y)$
$y = c .. d$

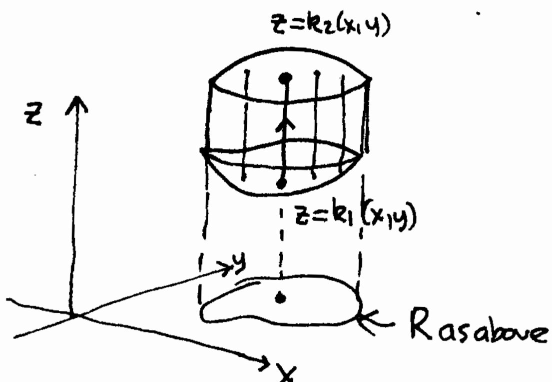
$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, \underbrace{dx\, dy}_{dA}$$

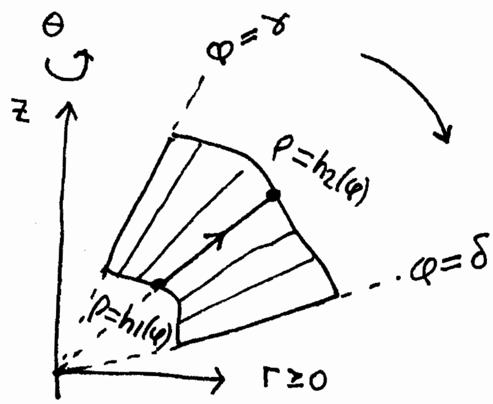
$r = g_1(\theta) .. g_2(\theta)$
$\theta = \alpha .. \beta$

$$\int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, \underbrace{r\, dr\, d\theta}_{dA}$$

3D regions E:

$$\iiint_E f(x,y,z)\, dV$$





$z = k_1(x,y) .. k_2(x,y)$

$$\iint_R \int_{k_1(x,y)}^{k_2(x,y)} f(x,y,z)\, dz\, dA$$

(includes cylindrical coords when R is a polar region)

$\rho = h_1(\varphi) .. h_2(\varphi)$
$\varphi = \gamma .. \delta$
$\theta = \alpha .. \beta$

$$\int_\alpha^\beta \int_\gamma^\delta \int_{h_1(\varphi)}^{h_2(\varphi)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, \underbrace{\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta}_{dV}$$

(this is special case of regions in r-z half-plane which revolve in the θ direction, i.e., about z-axis)