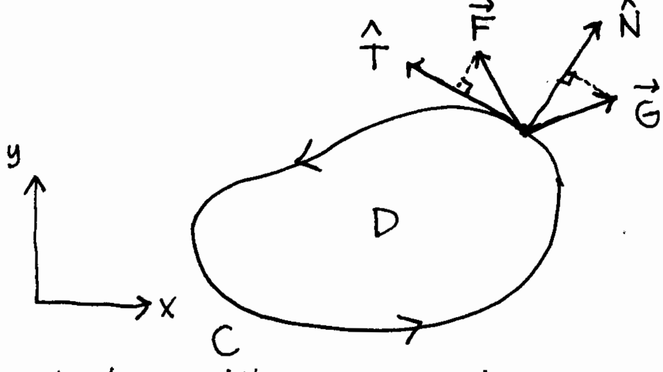
geometrical interpretation of Green' Theorem: Gauss and Stokes



■ Green's Theorem:

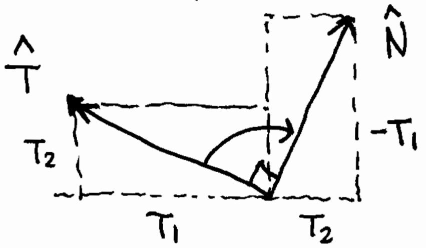
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$

$$\| $$

$$\oint_C \vec{F}\cdot\hat{T}\, ds$$ geometrical interpretation of LHS

closed curve with counterclockwise orientation (direction) enclosing a region D of the plane

let $\hat{N}$ be the outer unit normal (points away from D) obtained from the unit tangent $\hat{T}$ by a $90°$ clockwise rotation:

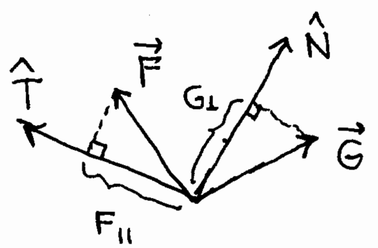


$$\hat{T} = \langle T_1, T_2\rangle \longrightarrow \hat{N} = \langle N_1, N_2\rangle = \langle T_2, -T_1\rangle$$

any vector field in the plane can be rotated pointwise by $90°$

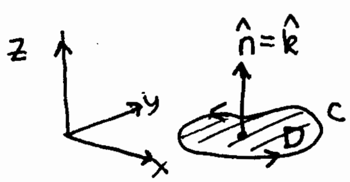
$$\vec{F} = \langle F_1, F_2\rangle \longrightarrow \vec{G} = \langle G_1, G_2\rangle = \langle F_2, -F_1\rangle$$

$$F_{\parallel} = \vec{F}\cdot\hat{T} = \vec{G}\cdot\hat{N} = G_{\perp}$$



(since rotation does not change the dot product between 2 vectors:
$\hat{N}\cdot\vec{G} = N_1G_1 + N_2G_2 = T_2F_2 + (-T_1)(-F_1) = T_1F_1 + T_2F_2 = \hat{T}\cdot\vec{F}$)

so tangential component of F equals the (outer) normal component of $\vec{G}$
(scalars: can be positive, negative or zero)



D is a surface in space.
its upward normal $\hat{n}$ obeys the righthand rule (RHR): curl fingers in direction of C, thumb points upwards

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \hat{k}\cdot(\vec{\nabla}\times\vec{F}) = \hat{k}\cdot \text{curl}\,\vec{F} = \hat{n}\cdot\text{curl}\,\vec{F}$$

■ Stoke's theorem version:

$$\oint_c \vec{F}\cdot\hat{T}\, ds = \iint_D (\text{curl}\,\vec{F})\cdot\hat{n}\, dA$$

integral of tangential component of vector field around closed loop equals the integral of the RHR normal component of its curl over the surface enclosed by the curve

$$\|$$

$$\frac{\partial G_1}{\partial x} - \frac{\partial(-G_2)}{\partial y} = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} = \vec{\nabla}\cdot\vec{G} = \text{div}\, G$$

■ Gauss's law version

$$\oint_C \vec{G}\cdot\hat{N}\, ds = \iint_D \text{div}\, G\, dA$$

integral of the outer normal component of a vector field around a closed loop equals the integral of its divergence over the interior of the loop