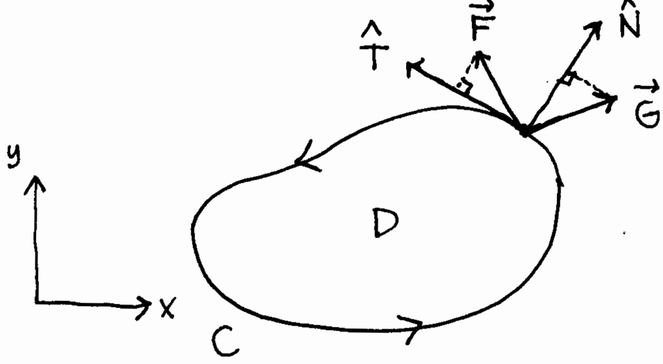
# geometrical interpretation of Green' Theorem: Gauss and Stokes




- **Green's Theorem:**

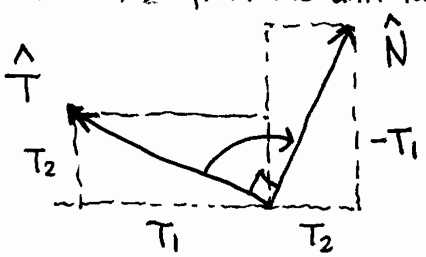
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\, dA$$

$$\parallel$$

$$\oint_C \vec{F}\cdot\hat{T}\, ds$$ geometrical interpretation of LHS

closed curve with counterclockwise orientation (direction) enclosing a region D of the plane

let $\hat{N}$ be the outer unit normal (points away from D) obtained from the unit tangent $\hat{T}$ by a $90^\circ$ clockwise rotation:

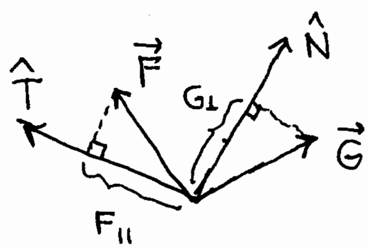



$$\hat{T}=\langle T_1,T_2\rangle \longrightarrow \hat{N}=\langle N_1,N_2\rangle=\langle T_2,-T_1\rangle$$

any vector field in the plane can be rotated pointwise by $90^\circ$

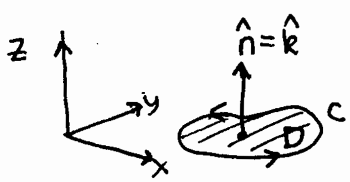
$$\vec{F}=\langle F_1,F_2\rangle \longrightarrow \vec{G}=\langle G_1,G_2\rangle=\langle F_2,-F_1\rangle$$

$$F_{\parallel}=\vec{F}\cdot\hat{T}=\vec{G}\cdot\hat{N}=G_{\perp}$$




(since rotation does not change the dot product between 2 vectors: $\hat{N}\cdot\vec{G}=N_1G_1+N_2G_2=T_2F_2+(-T_1)(-F_1)=T_1F_1+T_2F_2=\hat{T}\cdot\vec{F}$)

so tangential component of F equals the (outer) normal component of $\vec{G}$ (scalars: can be positive, negative or zero)




D is a surface in space.
its upward normal $\hat{n}$ obeys the righthand rule (RHR): curl fingers in direction of C, thumb points upwards

$$\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=\hat{k}\cdot(\vec{\nabla}\times\vec{F})=\hat{k}\cdot \text{curl}\,\vec{F}$$
$$=\hat{n}\cdot\text{curl}\,\vec{F}$$

- **Stoke's theorem version:**

$$\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_D (\text{curl}\,\vec{F})\cdot\hat{n}\,dA$$

integral of tangential component of vector field around closed loop equals the integral of the RHR normal component of its curl over the surface enclosed by the curve

$$\parallel$$

$$\frac{\partial G_1}{\partial x}-\frac{\partial(-G_2)}{\partial y}=\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}=\vec{\nabla}\cdot\vec{G}=\text{div}\,G$$

- **Gauss's law version**

$$\oint_C \vec{G}\cdot\hat{N}\,ds = \iint_D \text{div}\,G\,dA$$

integral of the outer normal component of a vector field around a closed loop equals the integral of its divergence over the interior of the loop

## divergence and curl

$$\vec{F} = \langle F_1, F_2, F_3 \rangle$$

$$\vec{\nabla} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle$$

$$\vec{\nabla} \cdot \vec{F} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \cdot \langle F_1, F_2, F_3 \rangle = \boxed{\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \text{div}\,\vec{F}}$$

$$\vec{\nabla} \times \vec{F} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \times \langle F_1, F_2, F_3 \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \boxed{\left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \underbrace{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}} \right\rangle = \text{curl}\,\vec{F}}$$

$$\hat{k} \cdot \text{curl}\,\vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

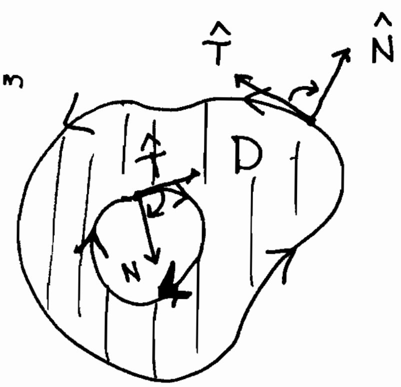
If $F_3 = 0$ $\quad \text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$

combinations which enter Greens Theorem for 2-d vector fields

---

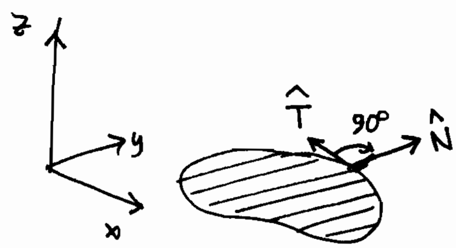
## Right Hand Rule : Thumbs Down!

2d Greens Thm with hole




$\hat{T}$ and outer $\hat{N}$ always related by clockwise $90°$ rotation

Within 3d space, $F_3 = 0$ 2d vector:




$\vec{F} \to -\hat{k} \times \vec{F}$ performs clockwise rotation in x-y plane

$$-\hat{k} \times \vec{F} = \begin{vmatrix} i & j & k \\ 0 & 0 & -1 \\ F_1 & F_2 & 0 \end{vmatrix} = \langle F_2, -F_1, 0 \rangle$$

thumb points down in space, fingers curl in the clockwise direction in x-y plane