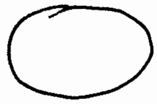
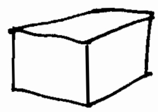
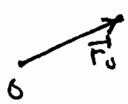
distributions of stuff: density functions (for mass, charge, probability, whatever)

"homogeneous" (= uniform) distribution of a total quantity Q of stuff
over an interval, planar region, volume

	measure	region	density $= \rho$		amount of stuff
1d	L	———	stuff/length	Q/L	in length ΔL : $\frac{Q}{L}\Delta L$
2d	A		stuff/area	Q/A	in area ΔA : $\frac{Q}{A}\Delta A$
3d	V		stuff/volume	Q/V	in volume ΔV : $\frac{Q}{V}\Delta V$

GENERALIZE TO NONUNIFORM "nonhomogeneous" DISTRIBUTION

nonconstant linear/planar/volume density function $\rho(x)$, $\rho(x,y)$, $\rho(x,y,z)$



$$\rho(\vec{r}_0) = \lim_{\vec{r}\to\vec{r}_0} \frac{\Delta(\text{stuff})}{\Delta(\text{length/area/volume})}$$

limiting ratio at a point defines the pointwise density

get total stuff by integrating density over region

$$Q = \int_{[a,b]} \underbrace{\rho(x)\,dx}_{dQ} \quad , \quad \iint_R \underbrace{\rho(x,y)\,dA}_{dQ} \quad , \quad \iiint_E \underbrace{\rho(x,y,z)\,dV}_{dQ}$$

$dQ =$ differential of stuff

If total stuff Q represents the total probability, which must be 1,
then $Q = 1$ is required.
Starting from any nonnegative function f defined on a region R, one can create a probability distribution function from it by "normalizing" it by dividing by its total integral over R:

$$\rho(\ldots) = \frac{f(\ldots)}{Q}$$

In 2-D: $Q = \iint_R f(x,y)\,dA$

$$\rho = f/Q = \frac{f(x,y)}{\iint_R f(x,y)\,dA}$$

Then $\iint_R \rho\, dA = 1$.

distributions of stuff (2)

Discrete average example

20 student class

distribution of grades

\# students who got grade of

4 : 80, 6 : 85, 5 : 90, 3 : 92, 1 : 98, 1 : 100

average grade : $\frac{4}{20}80 + \frac{6}{20}\cdot 85 + \frac{5}{20}\cdot 90 + \frac{3}{20}\cdot 92 + \frac{1}{20}\cdot 98 + \frac{1}{20}\cdot 100$

$= 87.7$

- weight each grade by the fraction of students with that grade — add up

grade that if all students had gotten would give same average grade

discrete → continuous
sum → integral

$$x_{avg} = \sum_{i=1}^{n} x_i f_i \rightarrow \int x f(x)\,dx$$

$$\sum f_i = 1 \rightarrow \int f(x)\,dx = 1$$

sum of all fractions is 1

we can average functions over a region weighted by the fraction of stuff

weighting function is fractional density :

$$\begin{cases} \frac{\rho(x)}{Q} : & \int \frac{\rho(x)}{Q}\,dx = \frac{\int \rho(x)\,dx}{Q} = 1 \\ \frac{\rho(x,y)}{Q} : & \iint \frac{\rho(x,y)}{Q}\,dA = \frac{\iint \rho(x,y)\,dA}{Q} = 1 \\ \frac{\rho(x,y,z)}{Q} : & \iiint \frac{\rho(x,y,z)}{Q}\,dV = \frac{\iiint \rho(x,y,z)\,dV}{Q} = 1 \end{cases}$$

stuff weighted averages are called "moments of the distribution" (when multiplied by Q)

<u>example</u>

$$\vec{r}_{avg} = \iiint \vec{r}\,\underbrace{\frac{\rho(x,y,z)}{Q}}_{\text{weighting factor for differential volume}}\,dV = \frac{\iiint \vec{r}\,\rho(x,y,z)\,dV}{\iiint \rho(x,y,z)\,dV}$$

$\vec{r}_{avg} = \langle \bar{x}, \bar{y}, \bar{z} \rangle$

this puts more weight on position vectors where more stuff is

(ditto for 1d, 2d examples)

called center of mass / charge / whatever

in probability distribution ($\rho \geq 0$), called "expected value of $\vec{r}$"

if $\rho = \rho_0$ constant, ρ cancels out of ratio → get geometric center of region (called "centroid")

just the average of the position vector over the region