Differentials

If we are only interested in the linear approximation CHANGE in the function value, then only the sum of the linear increments for each independent variable are needed:

$$\underbrace{f(x,y)}_{z} \approx \underbrace{f(x_0,y_0)}_{z_0} + f_x(x_0,y_0)\underbrace{(x-x_0)}_{dx} + f_y(x_0,y_0)\underbrace{(y-y_0)}_{dy}$$ linear approximation

convenient notation for increments. dx, dy, dz

$$\overbrace{z-z_0}^{dz} = f_x(x_0,y_0)\,(x-x_0) + f_y(x_0,y_0)\,(y-y_0)$$

$$dz = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy \equiv df(\underbrace{x_0,y_0,dx,dy}_{\text{function of 4 independent variables}})$$

Now no longer need to use subscript "0":

$$dz = f_x(x,y)dx + f_y(x,y)dy = df(x,y)$$ ← appropriate in function notation

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

appropriate if no "named function"

example 1. Area A of rectangle of sides x and y: $A = xy$ $\quad \frac{\partial A}{\partial x} = y$, $\frac{\partial A}{\partial y} = x$

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy = y\,dx + x\,dy$$

If sides change from x, y to $x+dx$, $y+dy$ then A changes from A to $A+dA$ (approximately), and the fractional change is $\frac{dA}{A} = \frac{y\,dx + x\,dy}{xy} = \frac{dx}{x} + \frac{dy}{y}$ namely the sum of the fractional changes in the dimensions.

example 2 The equivalent 4in×6in photo print format in Europe with the same dimension ratio $\frac{10}{15} = \frac{2}{3} = \frac{4}{6}$ is the 10cm × 15cm format.
If an actual print is measured with a millimeter ruler and found to have the dimensions 10cm × 15cm to within an error of ±0.2mm = ±0.02 cm, what is the computed error in their ratio? What is the percentage error?

$$R = \frac{x}{y} \qquad x = 10,\ |dx| \le 0.02 \qquad y = 15,\ |dy| \le 0.02$$

$$\frac{\partial R}{\partial x} = \frac{1}{y}, \quad \frac{\partial R}{\partial y} = \frac{\partial}{\partial y}(xy^{-1}) = -\frac{x}{y^2}, \quad dR = \frac{1}{y}dx - \frac{x}{y^2}dx = \frac{y\,dx - x\,dy}{y^2}$$

$$|dR| = \frac{|y\,dx - x\,dy|}{y^2} \le \frac{y|dx| + x|dy|}{y^2}$$ ← $|A+B| \le |A| + |B|$!

$$= \frac{15(0.2) + 10(0.2)}{15^2} = 0.2\left(\frac{25}{15^2}\right) = 0.2\left(\frac{5^2}{3^2 5^2}\right) = \frac{0.2}{9} \approx 0.022$$

The error in the ratio is about 0.022 (only 1 significant figure is warranted).

$$\frac{|dR|}{R} \le \frac{\frac{0.2}{9}}{2/3} = \frac{0.2}{6} = \frac{0.1}{3} \approx 0.033$$ about 3% error.

HW problem Calculate (using differentials) the absolute and percentage difference in the area of the exact 4in×6in print format relative to the 10 cm × 15 cm format. Compare them to the exact differences not using differentials.