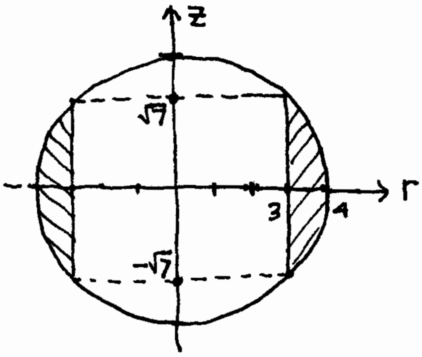
cylindrical and spherical regions of space

Drill a 3 unit radius hole through the center of a 4 unit radius sphere. How can we describe this solid using coordinates? Put the sphere at the origin and align the hole with the z-axis. Then it is a solid of revolution so we can look at its cross-section in the r-z plane.



sphere: $x^2+y^2+z^2=4^2 \leftrightarrow \rho^2=4^2 \rightarrow \rho=4$

cylinder: $x^2+y^2=3^2 \leftrightarrow r^2=3^2 \rightarrow r=3$

intersection: $3^2+z^2=4^2 \rightarrow z=\pm\sqrt{7}$

This is a ring shaped solid.
The range $0\le\theta\le 2\pi$ corresponds to revolving the right half of this diagram about the z-axis.

■ in terms of cylindrical coordinates r, θ, z:

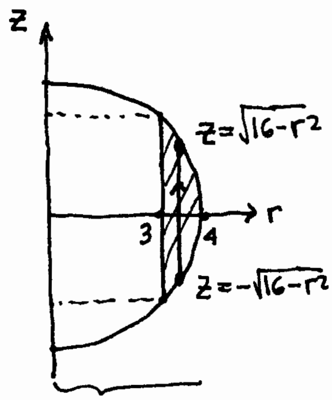
sphere: $r^2+z^2=4^2$ $\rightarrow$ intersection: $z^2=4^2-3^2=7,\ z=\pm\sqrt{7}$
cylinder: $r=3$

We have two ways of describing the solid's cross-section in the r-z halfplane

solve $r^2+z^2=4^2$ for z:

$3\le r\le 4$

$-\sqrt{16-r^2}\le z\le\sqrt{16-r^2}$
(lower) (upper)

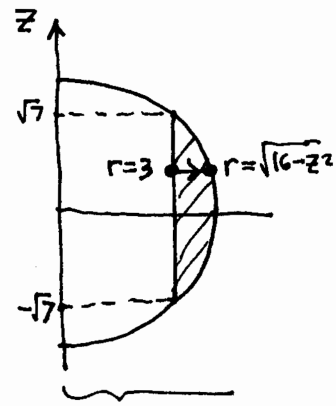


$\theta=\theta_0,\ r\ge 0$ cross-section

solve $r^2+z^2=4$ for r:

$-\sqrt{7}\le z\le\sqrt{7}$

$3\le r\le\sqrt{16-z^2}$
(inner) (outer)

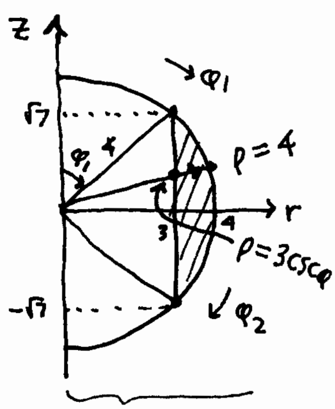


$\theta=\theta_0,\ r\ge 0$ cross-section

■ in terms of spherical coordinates ρ, φ, θ:

$\varphi_1\le\varphi\le\varphi_2$

$3\csc\varphi\le\rho\le 4$
(inner) (outer)



$\theta=\theta_0,\ r\ge 0$ cross-section

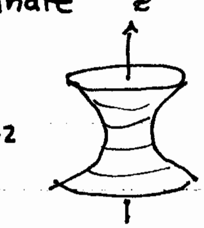
φ range:
$\cos\varphi_1=\frac{\sqrt{7}}{4} \rightarrow \varphi_1=\arccos\frac{\sqrt{7}}{4}$ ($\approx 48.6°$)
$\varphi_2=\pi-\varphi_1=\pi-\arccos\frac{\sqrt{7}}{4}$ ($\approx 131.4°$)

Now we need to describe the cylinder $r=3$ in terms of spherical coordinates using $r=\rho\sin\varphi,\ z=\rho\cos\varphi$ (see next page)
$3=\rho\sin\varphi \rightarrow \rho=3/\sin\varphi=3\csc\varphi$

cylindrical and spherical coordinate regions of space (2)

solid regions in space can often be described as between two graphs of functions of two of the three coordinates, as in the previous example, i.e. between two surfaces which can be described by expressing one coordinate as a function of the other two (even if it depends only on one of them).

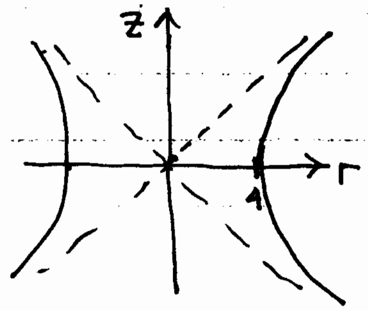
Here is another example of describing a surface in the three coordinate systems:



$$\underbrace{x^2+y^2}-z^2=16$$

surface of revolution since only depends on $x^2+y^2=r^2$

■ $r^2-z^2=16$ (cylindrical coords)



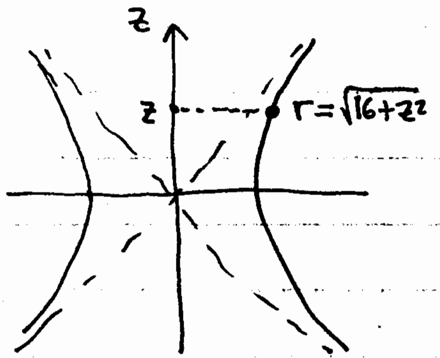
hyperbola in r-z plane; only has r-intercepts: $z=0 \to r=4$

so opens up about z-axis.

when r, z are very large compared to 16:

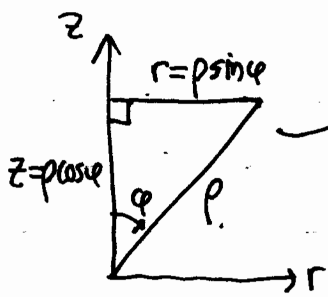
$r^2-z^2 \approx 0 \to z=\pm r$ (asymptotes)

It is natural to express $r \geq 0$ in terms of z:



$$\left.\boxed{\begin{array}{l} r=\sqrt{16+z^2} \\ -\infty \leq z \leq \infty \\ 0 \leq \theta \leq 2\pi \end{array}}\right\}$$ cyl coord description

■ We can also describe this in spherical coords, expressing $\rho \geq 0$ in terms of φ:



$$r^2-z^2=16$$

$$(\rho\sin\varphi)^2-(\rho\cos\varphi)^2=16$$

$$\rho^2\sin^2\varphi-\rho^2\cos^2\varphi=16$$

$$\rho^2(\sin^2\varphi-\cos^2\varphi)=16$$

better to express in terms of sin or cos

$$\rho^2=\frac{16}{\sin^2\varphi-\cos^2\varphi}=\frac{16}{(1-\cos^2\varphi)-\cos^2\varphi}=\frac{16}{1-2\cos^2\varphi}$$

allowed values of φ:

$1-2\cos^2\varphi \leq 0$

$\frac{1}{2} < \cos^2\varphi \to \frac{1}{\sqrt{2}} < |\cos\varphi|$

$\cos\varphi=\pm\frac{1}{\sqrt{2}} \to \varphi=\pi/4, \frac{3\pi}{4}$

(also clear from asymptotes $z=\pm r$ slope ± 1.)

sph coord description
$$\left\{\boxed{\begin{array}{l} \rho=\dfrac{4}{\sqrt{1-2\cos^2\varphi}} \\ \frac{\pi}{4} \leq \varphi < \frac{3\pi}{4} \\ 0 \leq \theta \leq 2\pi \end{array}}\right.$$