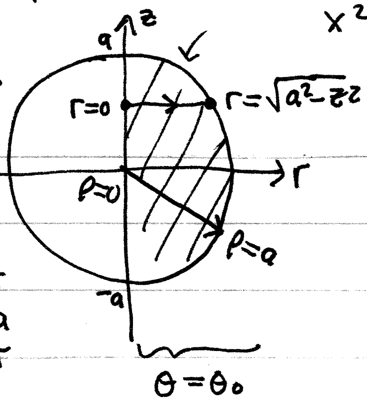
triple integrals in cylindrical and spherical coordinates

sphere

$x^2+y^2+z^2=a^2 \to r^2+z^2=a^2 \to r=\sqrt{a^2-z^2}$

radial variable limits:



$0\le\theta\le 2\pi$

$-a\le z\le a$

$0\le\varphi\le\pi$

$$V=\int_0^{2\pi}\int_{-a}^{a}\int_0^{\sqrt{a^2-z^2}} 1\, r\,dr\,dz\,d\theta$$

$$\frac{r^2}{2}\Big|_{r=0}^{r=\sqrt{a^2-z^2}} = \frac{a^2-z^2}{2}$$

$$\frac{a^2z-z^3/3}{2}\Big|_{z=-a}^{z=a} = \frac{a^2a-a^3/3}{2} - \frac{(a^2(-a)-(-a)^3/3)}{2}$$

$$= a^3\frac{(1-1/3)}{2} + a^3\frac{(1-1/3)}{2} = \frac{2a^3}{3}$$

$$= \int_0^{2\pi}\frac{2a^3}{3}\,d\theta = \frac{2a^3}{3}\theta\Big|_0^{2\pi} = \frac{2a^3}{3}(2\pi) = \frac{4\pi a^3}{3} \checkmark$$

$$V=\int_0^{2\pi}\int_0^{\pi}\int_0^{a} 1\ \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\frac{2a^3}{3}\,d\theta = \frac{2a^3}{3}(2\pi) = 4\pi a^3/3 \checkmark$$

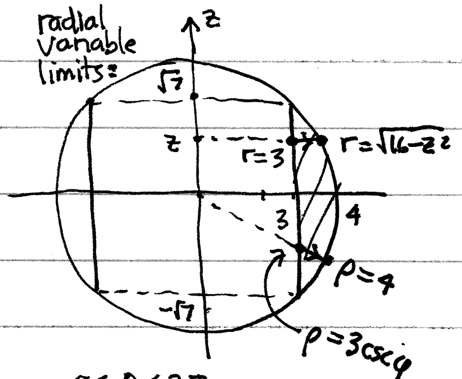
$$(\sin\varphi)\frac{\rho^3}{3}\Big|_{\rho=0}^{\rho=a} = \frac{a^3}{3}\sin\varphi$$

$$-\frac{a^3}{3}\cos\varphi\Big|_{\varphi=0}^{\varphi=\pi} = -\frac{a^3}{3}(-1)+\frac{a^3}{3}(1) = \frac{2a^3}{3}$$

sphere with hole

3 in radius hole in 4 in radius sphere

radial variable limits:



$0\le\theta\le 2\pi$

$-\sqrt7\le z\le\sqrt7$

$\arccos\frac{\sqrt7}{4}\le\varphi\le\pi-\arccos\frac{\sqrt7}{4}$

$$V=\int_0^{2\pi}\int_{-\sqrt7}^{\sqrt7}\int_3^{\sqrt{16-z^2}} 1\ r\,dr\,dz\,d\theta$$

$$\frac{r^2}{2}\Big|_{r=3}^{r=\sqrt{16-z^2}} = \frac{(16-z^2)-9}{2} = \frac{7-z^2}{2}$$

$$\frac{7z-z^3/3}{2}\Big|_{z=-\sqrt7}^{z=\sqrt7} = \frac{7\sqrt7-7\sqrt7/3}{2} - \frac{(7(-\sqrt7)-(-\sqrt7)^3/3)}{2}$$

$$= 7\sqrt7\frac{(1-1/3)}{2} + 7\sqrt7\frac{(1-1/3)}{2} = \frac{14\sqrt7}{3}$$

$$= \int_0^{2\pi}\frac{14\sqrt7}{3}\,d\theta = \frac{14\sqrt7}{3}(2\pi) = \frac{28\pi\sqrt7}{3}$$

$$V=\int_0^{2\pi}\int_{\arccos\frac{\sqrt7}{4}}^{\pi-\arccos\sqrt7/4}\int_{3\csc\varphi}^{4} 1\ \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

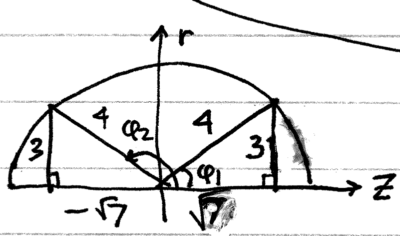
$$\frac{\rho^3}{3}\sin\varphi\Big|_{\rho=3\csc\varphi}^{\rho=4} = \frac{4^3}{3}\sin\varphi - \frac{3^3}{3}\underbrace{\csc^3\varphi\sin\varphi}_{\csc^2\varphi}$$

standard integral:
$\int\csc^2\varphi\,d\varphi = -\cot\varphi + C$

$$-\frac{4^3}{3}\cos\varphi + 9\cot\varphi\Big|_{\varphi=\arccos\sqrt7/4}^{\varphi=\pi-\arccos\sqrt7/4} = -\frac{4^3}{3}\left(-\frac{\sqrt7}{4}\right)+9\left(-\frac{\sqrt7}{3}\right)+\frac{4^3}{3}\left(\frac{\sqrt7}{4}\right)-9\left(\frac{\sqrt7}{3}\right)$$

$$= \frac{32}{3}\sqrt7-6\sqrt7 = \frac{14}{3}\sqrt7$$

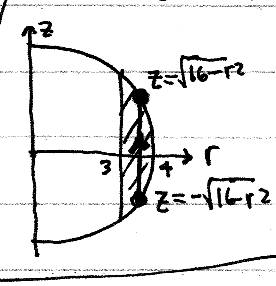
$$= \int_0^{2\pi}\frac{14\sqrt7}{3}\,d\theta = \frac{14\sqrt7}{3}(2\pi) = \frac{28\sqrt7\pi}{3} \checkmark$$



$\cos\varphi_2 = -\sqrt7/4$ $\quad\cos\varphi_1 = \sqrt7/4$

$\cot\varphi_2 = -\sqrt7/3$ $\quad\cot\varphi_1 = \sqrt7/3$

But the easiest way is:



$$V=\int_0^{2\pi}\int_3^4\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} 1\ r\,dz\,dr\,d\theta$$

$$z\Big|_{-\sqrt{\ }}^{\sqrt{\ }} = 2\sqrt{16-r^2}$$

$$\int 2\sqrt{\underbrace{16-r^2}_{u}}\ r\,dr \to -du \quad = \int u^{1/2}(-du)$$

$$= -\frac{2}{3}u^{3/2} = -\frac{2}{3}(16-r^2)^{3/2}$$

$$V=\int_0^{2\pi} -\frac{2}{3}(16-r^2)^{3/2}\Big|_3^4\,d\theta$$

$$=(2\pi)\,\frac{2}{3}\,7^{3/2} = \frac{28\sqrt7\pi}{3} \checkmark$$