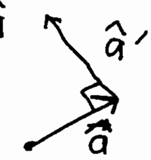
Geometry of Curves

preliminaries:

chain rule : $\frac{dQ}{ds} = \frac{dQ/dt}{ds/dt} = \frac{Q'}{s'}$

length direction decomposition: $\vec a \to |\vec a|$, $\hat a = \frac{\vec a}{|\vec a|}$ $\to$ $\vec a = |\vec a|\,\hat a$

unit vector derivative : $\hat a \cdot \hat a' = 0$



$\vec r = \vec r(t) = \langle x(t), y(t), z(t)\rangle$

$\downarrow D$

$\vec r' \to |\vec r'| = s'$, $\quad \hat T = \frac{\vec r'}{|\vec r'|} \to \vec r' = s'\hat T \xrightarrow{D} \vec r'' = s''\hat T + s'\hat T'$

$\vec r' \times \vec r'' = (s'\hat T) \times (s''\hat T + s'\hat T') = s'^2\, \hat T \times \hat T'$

$|\vec r' \times \vec r''| = s'^2 \underbrace{|\hat T \times \hat T'|}_{|\hat T||\hat T'|\sin\frac{\pi}{2}} = s'^2 \underbrace{|\hat T'|}_{\kappa s'} = s'^3 \kappa$

$$\boxed{\kappa = \frac{|\vec r' \times \vec r''|}{s'^3} = \frac{|\vec r' \times \vec r''|}{|\vec r'|^3}}$$

$\hat T' \to \hat N = \frac{\hat T'}{|\hat T'|}$

$$\left[\hat B = \frac{\vec r' \times \vec r''}{|\vec r' \times \vec r''|} = \hat T \times \hat N\right]$$

$\frac{d\hat T}{ds} = \frac{\hat T'}{s'} \to \left|\frac{d\hat T}{ds}\right| = \boxed{\frac{|\hat T'|}{s'} = \kappa = \frac{|\hat T'|}{|\vec r'|}} = \frac{1}{\rho}$

HELIX EXAMPLE

$\vec r = \langle a\cos t, a\sin t, bt\rangle \quad a, b > 0$

$\vec r' = \langle -a\sin t, a\cos t, b\rangle \xrightarrow{\text{or}} \vec r'' = \langle -a\cos t, -a\sin t, 0\rangle$

$|\vec r'| = \sqrt{a^2s^2 + a^2c^2 + b^2} = \sqrt{a^2+b^2} = s'$

$\hat T = \frac{1}{\sqrt{a^2+b^2}}\langle -a\sin t, a\cos t, b\rangle$

$\hat T' = \frac{1}{\sqrt{a^2+b^2}}\langle -a\cos t, -a\sin t, 0\rangle$

$|\hat T'| = \frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2c^2 + a^2s^2} = \frac{a}{\sqrt{a^2+b^2}}$

$\kappa = \frac{|\hat T'|}{s'} = \frac{a/\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2} = \frac{1}{\rho}$

$\rho = \frac{a^2+b^2}{a} = a + \frac{b^2}{a} > a$

$\hat N = \hat T' / |\hat T'| = \frac{1/\sqrt{a^2+b^2}}{a/\sqrt{a^2+b^2}}\langle \cdots \rangle = \underbrace{\langle -\cos t, -\sin t, 0\rangle}$ horizontal, points towards z-axis

$\hat B = \hat T \times \hat N = \frac{1}{\sqrt{a^2+b^2}}\begin{vmatrix} i & j & k \\ -as & ac & b \\ -c & -s & 0 \end{vmatrix} = \frac{1}{\sqrt{a^2+b^2}}\langle 0+bs, -bc-0, a(s^2+c^2)\rangle$

$= \frac{1}{\sqrt{a^2+b^2}}\langle b\sin t, -b\cos t, a\rangle$

$\vec r' \times \vec r'' = \begin{vmatrix} i & j & k \\ -as & ac & b \\ -ac & -as & 0 \end{vmatrix} = \langle 0+abs, -abc-0, a^2(s^2+c^2)\rangle$

$= \langle abs, -abc, a^2\rangle$

$= a\langle bs, -bc, a\rangle$

$|\vec r' \times \vec r''| = a\sqrt{b^2s^2 + b^2c^2 + a^2} = a\sqrt{a^2+b^2}$

$\kappa = \frac{|\vec r' \times \vec r''|}{|\vec r'|^3} = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2}$

$\hat B = \frac{\vec r' \times \vec r''}{|\vec r' \times \vec r''|} = \frac{a}{a\sqrt{a^2+b^2}}\langle bs, -bc, a\rangle$

$= \frac{1}{\sqrt{a^2+b^2}}\langle b\sin t, -b\cos t, a\rangle$