space curvature and acceleration

arclength $s = \int_{t_0}^{t} |\vec{r}\,'(t)|\,dt \leftrightarrow s' = \frac{ds}{dt} = |\vec{r}\,'(t)|$ (speed)

position: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

velocity: $\vec{v}(t) = \vec{r}\,'(t)$ (tangent) $\longrightarrow$ length: $|\vec{r}\,'(t)|$

$\searrow$ direction: $\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ (unit tangent)

$$\left.\right\} \quad \vec{r}\,'(t) = |\vec{r}\,'(t)|\,\hat{T}(t) = s'(t)\,\hat{T}(t)$$

acceleration $\vec{a}(t) = \vec{r}\,''(t) = \frac{d}{dt}\left(s'(t)\hat{T}(t)\right) \underset{\text{prod rule}}{=} s''(t)\,\hat{T}(t) + s'(t)\,\underbrace{\hat{T}'(t)}_{\text{what's this?}}$

$\hat{T}$ is a unit vector so it can only rotate $\Rightarrow$ $\hat{T}'$ is $\perp$ to $\hat{T}$

$$\left\{ \begin{array}{l} \frac{d}{dt}\left(\hat{T}\cdot\hat{T} = 1\right) \xrightarrow[\text{rule}]{\text{prod}} \hat{T}'\cdot\hat{T} + \hat{T}\cdot\hat{T}' = 0 \\ \qquad 2\hat{T}\cdot\hat{T}' = 0 \rightarrow \hat{T}\cdot\hat{T}' = 0 \\ \hat{T}' \text{ gives direction in which tip of } \hat{T} \text{ is rotating} \end{array} \right.$$

so $\hat{N}(t) = \frac{\hat{T}'(t)}{|\hat{T}'(t)|} = \frac{\frac{d}{dt}\left(\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\right)}{\left|\frac{d}{dt}\left(\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\right)\right|}$ quotient rule + divide by length calculation $=$ unit normal is direction in which $\hat{T}$ is rotating: $\hat{N}(t)\cdot\hat{T}(t) = 0$

But only arclength derivatives produce information about the path independent of how fast one moves along it.

CHAIN RULE allows us to compute such derivatives without having the curve parametrized by arclength s: $\quad \frac{df}{ds} = \frac{df/dt}{ds/dt} = \frac{f'}{s'} = \frac{f'}{|\vec{r}\,'|}$.

$$\frac{d\vec{r}}{ds} = \frac{\vec{r}\,'}{s'} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \hat{T} \quad \text{(as before)}$$

$$\frac{d^2\vec{r}}{ds^2} = \frac{d\hat{T}}{ds} = \underbrace{\left|\frac{d\hat{T}}{ds}\right|}_{\text{length}} \underbrace{\widehat{\frac{d\hat{T}}{ds}}}_{\text{direction}} = \kappa\hat{N}$$

$\kappa \geq 0$ curvature; $\hat{N}$ unit normal $= \frac{d\hat{T}/ds}{|d\hat{T}/ds|} = \frac{\hat{T}'}{|\hat{T}'|}$ (as before)

chain rule: $\frac{d\hat{T}}{ds} = \frac{d\hat{T}/dt}{ds/dt} = \frac{\hat{T}'}{s'}$ — s' cancels out

$$\rightarrow \boxed{\hat{T}' = s'\kappa\hat{N}} \quad \text{substitute above}$$

radius of curvature $\kappa(t) = \frac{1}{\rho(t)}$

$$\boxed{\begin{array}{l}\text{decomposition}\\ \text{of}\\ \text{acceleration}\end{array}} \quad \underbrace{\vec{r}\,''(t)}_{\vec{a}(t)} = \underbrace{s''(t)\,\hat{T}(t)}_{a_{\hat{T}}} + \underbrace{s'(t)^2\,\kappa(t)\,\hat{N}(t)}_{a_{\hat{N}}}$$

$a_{\hat{T}} =$ linear acc. along direction of motion "tangential acc."

$a_{\hat{T}} = \vec{a}\cdot\hat{T}$

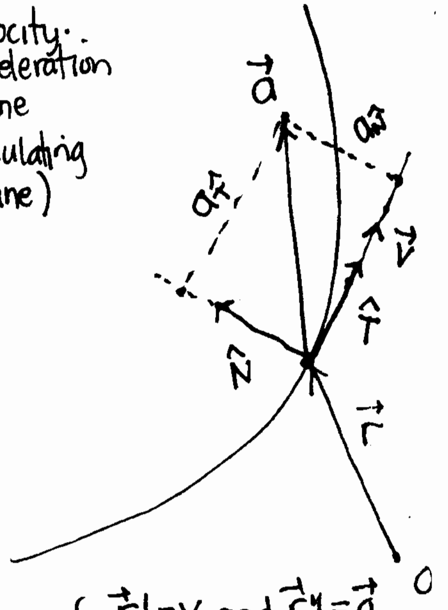
$a_{\hat{N}} =$ acc. perp to direction of motion $= \frac{s'(t)^2}{\rho(t)}$ $\left(\text{"}\frac{v^2}{r}\text{"}\right)$

"normal acceleration" "centripetal acc."

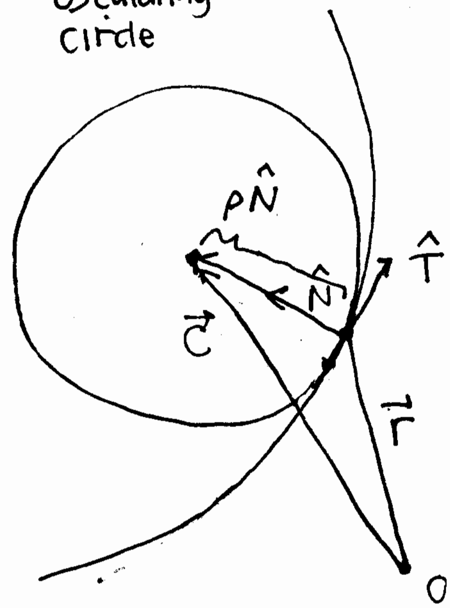
$a_{\hat{N}} = \vec{a}\cdot\hat{N}$

space curvature and acceleration (2)

velocity-acceleration plane (osculating plane)



osculating circle



plane of $\vec{r}\,' = v$ and $\vec{r}\,'' = \vec{a}$
or of $\hat{T}$ and $\hat{N}$

normal: $\vec{r}\,' \times \vec{r}\,''$ or $\underbrace{\hat{T} \times \hat{N}}$

$\hat{B}$ = binormal = unit normal to vel-acc plane

$$= \frac{\vec{r}\,' \times \vec{r}\,''}{|\vec{r}\,' \times \vec{r}\,''|}$$

center is a distance ρ along $\hat{N}$ from the curve:

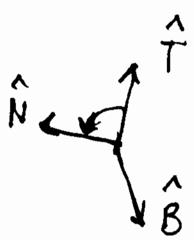
$$\vec{C} = \vec{r} + \rho\hat{N}$$

(Note $|\hat{B}| = \underbrace{|\hat{T}|}_{1}\,\underbrace{|\hat{N}|}_{1}\,\underbrace{\sin\frac{\pi}{2}}_{1} = 1$
unit vector perp to $\hat{T}$ and $\hat{N}$)

But $\underbrace{\vec{r}\,'}_{s'\hat{T}} \times \underbrace{\vec{r}\,''}_{s''\hat{T} + s'^2 \kappa N} = s'\hat{T} \times (s''\hat{T} + s'^2\kappa N) = s's''\underbrace{\hat{T} \times \hat{T}}_{0} + s'^3\kappa\,\underbrace{\hat{T} \times N}_{\hat{B}}$

$$|\vec{r}\,' \times \vec{r}\,''| = s'^3 \kappa\,\underbrace{|\hat{T} \times \hat{N}|}_{1} \rightarrow \boxed{\vec{\kappa} = \frac{|r' \times r''|}{s'^3} = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}}$$ 1-2-3 rule

avoids quotient rule computation



right hand rule:
$$\hat{N} = \hat{B} \times \hat{T}$$

compute $\hat{N}$ by cross-product instead of $\hat{T}'$ derivative calculation too.

space curvature and acceleration (3)

twisted cubic example (rescaled to make perfect square ↓)

$$\vec{r} = \langle 2t, t^2, t^3/3 \rangle$$

$$\vec{r}\,' = \langle 2, 2t, t^2 \rangle \quad\to\quad |\vec{r}\,'| = \sqrt{4+4t^2+t^4} = \sqrt{(t^2+2)^2} = t^2+2$$

$$\to\quad \hat{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \frac{\langle 2, 2t, t^2 \rangle}{t^2+2}$$

$$\vec{r}\,'' = \langle 0, 2, 2t \rangle = 2\langle 0,1,t \rangle$$

$$\vec{r}\,' \times \vec{r}\,'' = 2\,\langle 2,2t,t^2 \rangle \times \langle 0,1,t \rangle = 2 \begin{vmatrix} i & j & k \\ 2 & 2t & t^2 \\ 0 & 1 & t \end{vmatrix} = 2\langle 2t^2-t^2, 0-2t, 2-0 \rangle = 2\langle t^2, -2t, 2 \rangle$$

$$|\vec{r}\,' \times \vec{r}\,''| = 2\sqrt{t^4+4t^2+4} = 2(t^2+2) \quad \text{(as before)}$$

$$K = \frac{|r' \times r''|}{|r'|^3} = \frac{2(t^2+2)}{(t^2+2)^3} = \frac{2}{(t^2+2)^2} \to \rho = \frac{(t^2+2)^2}{2}$$

$$\hat{B} = \frac{\vec{r}\,' \times \vec{r}\,''}{|r' \times r''|} = \frac{2\langle t^2, -2t, 2 \rangle}{2\ (t^2+2)} = \frac{\langle t^2, -2t, 2 \rangle}{t^2+2}$$

$$\hat{N} = \hat{B} \times \hat{T} = \frac{1}{(t^2+2)}\frac{1}{(t^2+2)} \begin{vmatrix} i & j & k \\ t^2 & -2t & 2 \\ 2 & 2t & t^2 \end{vmatrix} = \frac{1}{(t^2+2)^2}\langle \underbrace{-2t^3-4t}_{-2t(t^2+2)}, \underbrace{4-t^4}_{(2-t^2)(2+t^2)}, \underbrace{2t^3+4t}_{2t(t^2+2)} \rangle$$

$$= \frac{\langle -2t, 2-t^2, 2t \rangle}{t^2+2}$$

or $\hat{T} = \dfrac{\langle 2,2t,t^2 \rangle}{t^2+2}$

$$\hat{T}\,' = \frac{(t^2+2)\langle 0,2,2t \rangle - \langle 2,2t,t^2 \rangle (2t)}{(t^2+2)^2}$$

$$= \frac{\langle 0, 2t^2+4, 2t^3+4t \rangle - \langle 4t, 4t^2, 2t^3 \rangle}{(t^2+2)^2}$$

$$= \frac{\langle -4t, 4-2t^2, 4t \rangle}{(t^2+2)^2} = \frac{2}{(t^2+2)^2}\langle -2t, 2-t^2, 2t \rangle$$

$$\text{so } \hat{N} = \hat{T}\,'/|\hat{T}\,'| = \frac{\langle -2t, 2-t^2, 2t \rangle}{\sqrt{4t^2+(4-4t^2+t^4)+4t^2}}$$

$$\sqrt{(t^2+2)^2} = t^2+2$$

at $t=0$:

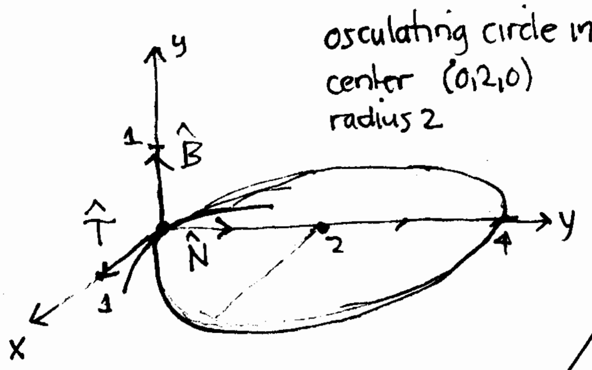
$\vec{r}(0) = \langle 0,0,0 \rangle$

$\hat{T}(0) = \dfrac{\langle 2,0,0 \rangle}{2} = \langle 1,0,0 \rangle = \hat{i}$

$\hat{N}(0) = \dfrac{\langle 0,2,0 \rangle}{2} = \langle 0,1,0 \rangle = \hat{j}$

$\hat{B}(0) = \dfrac{\langle 0,0,2 \rangle}{2} = \langle 0,0,1 \rangle = \hat{k}$

$K(0) = \dfrac{2}{2^2} = \dfrac{1}{2} \to \rho(0) = 2$



osculating circle in xy plane
center $(0,2,0)$
radius 2

acceleration decomposition:

$$\vec{a} = \vec{r}\,'' = 2\langle 0,1,t \rangle$$

$$a_{\hat{T}} = \hat{T} \cdot \vec{a} = 2\,\frac{\langle 2,2t,t^2 \rangle \cdot \langle 0,1,t \rangle}{t^2+2}$$

$$= \frac{2(2t+t^3)}{(t^2+2)} = \frac{2t(t^2+2)}{(t^2+2)} = 2t$$

$$a_{\hat{N}} = \hat{N} \cdot \vec{a} = 2\,\frac{\langle -2t, 2-t^2, 2t \rangle \cdot \langle 0,1,t \rangle}{(t^2+2)}$$

$$= \frac{2(2-t^2+2t^2)}{(t^2+2)} = \frac{2(t^2+2)}{(t^2+2)} = 2$$

$$\vec{a} = a_{\hat{T}}\hat{T} + a_{\hat{N}}\hat{N} = 2t\,\hat{T} + 2\,\hat{N}.$$