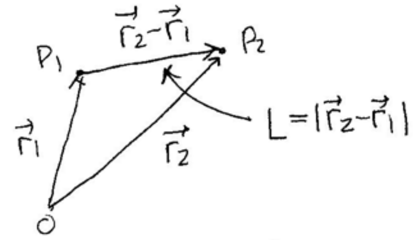
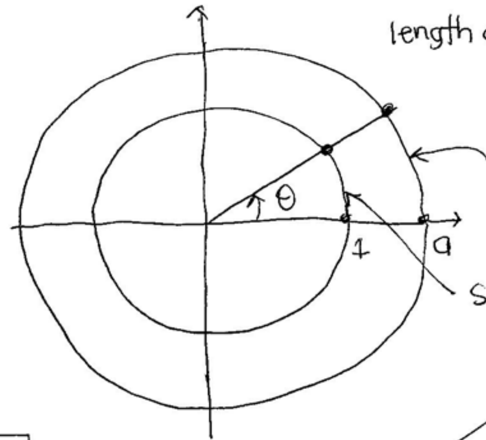
Arclength



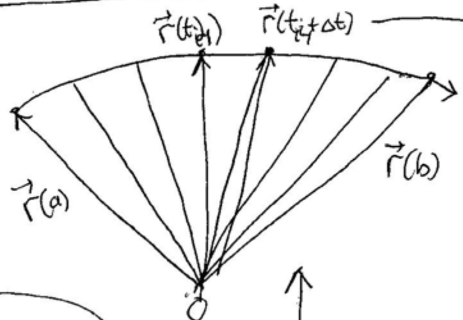
length of straight line segment = length difference vector



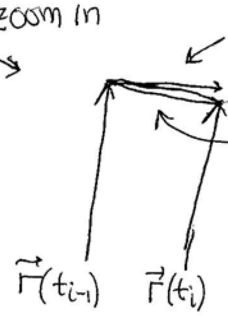
length of an arc "arclength" = radius times angle

arc = part of circle

$\boxed{\text{generalize to any curve segment } \vec{r}(t) \text{ for } a \le t \le b:}$



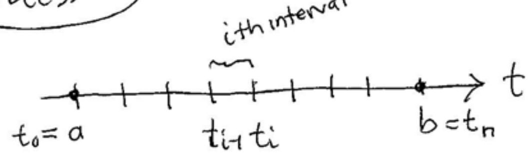
zoom in



geometric interpretation of derivative

secant line difference

Riemann sum integration process



equally divide up interval $\Delta t = t_i - t_{i-1} = \frac{b-a}{n}$

choose value in intervals: $t_{i-1} \le t_i^* \le t_i$

$$\Delta L_i \approx |\vec{r}(t_{i-1}+\Delta t) - \vec{r}(t_{i-1})| \approx |\vec{r}\,'(t_{i-1})|\Delta t \approx |\vec{r}\,'(t_i^*)|\Delta t$$

approximate ΔL_i by length of tangent for some value t_i^* in each interval times Δt, sum them up:

$$L \approx \sum_{i=1}^{n} \Delta L_i = \sum_{i=1}^{n} |\vec{r}\,'(t_i^*)|\Delta t$$

$\downarrow$ take limit

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |\vec{r}\,'(t_i^*)|\Delta t$$

$$\equiv \int_a^b |\vec{r}\,'(t)|\, dt$$

definition of Riemann integral

$\boxed{\text{Find arclength of curve by integrating the length of the tangent vector}}$

Example. Helix: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle \to \vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

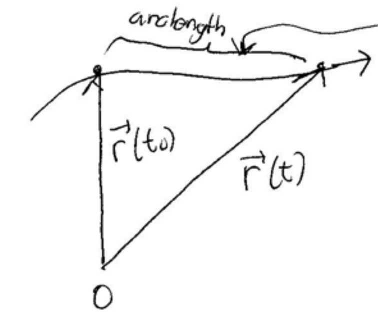
one loop: $0 \le t \le 2\pi \longrightarrow L = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\, t \Big|_0^{2\pi} = 2\pi\sqrt{2}$

Fact: $\int_a^b |\vec{r}\,'(t)|\, dt = \int_a^b \sqrt{\underbrace{x'(t)^2 + y'(t)^2 + z'(t)^2}}\, dt$

unless this is a perfect square so that the sqrt is undone, very few such integrands can be integrated exactly!
for helix it is constant so trivial

Arclength function

Fix reference point, measure arclength up to all other points on curve.



$$s(t) = \text{"} \int_{t_0}^{t} |\vec{r}\,'(t)|\,dt \text{"} = \int_{t_0}^{t} |\vec{r}\,'(u)|\,du$$

"dummy integration variable" u to avoid confusion with upper limit t

$$\underbrace{s'(t)}_{= \frac{ds(t)}{dt}} = \frac{d}{dt}\left(\int_{t_0}^{t} |\vec{r}\,'(u)|\,du\right) = |\vec{r}\,'(t)|$$

Recall $\frac{d}{dx}\left(\int_a^x f(u)\,du\right) = f(x)$

$\boxed{\text{rate of change of arclength} = \text{length of tangent vector}}$

physics of motion: "time rate of change of distance traveled equals speed"

$\frac{ds}{dt} = |\vec{r}\,'(t)| \longrightarrow ds = |\vec{r}\,'(t)|\,dt$ "differential of arclength"

arclength parametrization

given a parametrized curve $\vec{r}(t)$, one can reparametrize it by the arclength from some reference point if one can evaluate the integral for $s(t)$, then suceed in solving this for t as a function of s (rarely possible).

example. Helix. $s = \int_0^t \sqrt{2}\,du = \sqrt{2}\Big|_0^t = \sqrt{2}\,t \rightarrow t = s/\sqrt{2}$ substitute:

$\vec{r} = \langle \cos t, \sin t, t\rangle = \langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\rangle$ new parametrized curve with same path

$\frac{d\vec{r}}{ds} = \langle -\sin\frac{s}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle = \frac{1}{\sqrt{2}}\langle -\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, 1\rangle$ new tangent vector

$\left|\frac{d\vec{r}}{ds}\right| = \frac{1}{\sqrt{2}}\sqrt{\sin^2\frac{s}{\sqrt{2}} + \cos^2\frac{s}{\sqrt{2}} + 1} = \frac{1}{\sqrt{2}}\sqrt{2} = 1$ unit vector

If we introduce the unit tangent $\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ giving the direction of the tangent vector,

then $\boxed{\frac{d\vec{r}}{ds} = \hat{T}}$ in the arclength parametrization the tangent vector is automatically a unit vector.