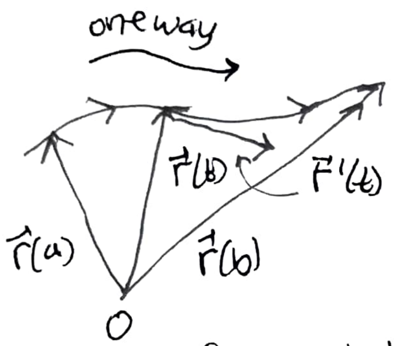
Vector line integrals (1)

The line integral of a vector field along a curve REQUIRES an $\boxed{\text{ORIENTED CURVE}}$

$C: \vec{r} = \vec{r}(t),\ t = a..b$ such that $|\vec{r}\,'(t)| \neq 0 \Leftrightarrow \vec{r}\,'(t) = \vec{0}$ for any t

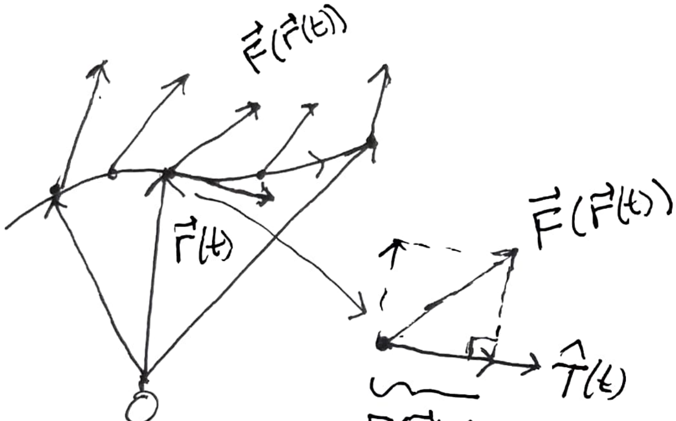
when $\vec{r}\,'(t_0) = \vec{0}$, $\hat{T}(t_0)$ is undefined, $\hat{T}(t)$ can change direction across the point $\vec{r}(t_0)$

OR $\boxed{\text{DIRECTED CURVE}}$



direction of increasing t

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$ defines local direction at each point of curve



$F(\vec{r}(t))_{\parallel} = \vec{F}(\vec{r}(t)) \cdot \hat{T}(t)$

tangential component of $\vec{F}$ along curve

Integrate tangential component of $\vec{F}$ along C: with respect to differential of arclength:

$$ds(t) = |\vec{r}\,'(t)|\, dt$$

$$\text{"}\int_C \vec{F}\cdot d\vec{r}\,\text{"} \equiv \int_a^b \underbrace{\vec{F}(\vec{r}(t)) \cdot \hat{T}(t)}_{F_{\parallel}}\ \underbrace{|\vec{r}\,'(t)|\, dt}_{ds}$$

troublesome sqrt expression cancels out! vector line integrals easier to evaluate than scalar line integrals!

$$\equiv \int_C F_{\parallel}\, ds$$

$\vec{r}\,'(t) = |\vec{r}\,'(t)|\, \hat{T}(t)$!

$$= \boxed{\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt}$$

($\vec{r}\,' \to -\vec{r}\,'$ changes sign of integral)

symbolic manipulation: $\int_C \vec{F} \cdot \hat{T}\, ds$

$$d\vec{r} \equiv \hat{T}\, ds = \underbrace{\hat{T}\, |\vec{r}\,'|}_{\vec{r}\,'} dt = \vec{r}\,' dt = \frac{d\vec{r}}{dt} dt \ !$$

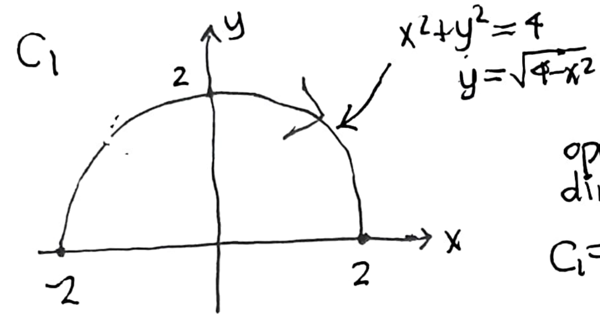
vector form preferable $\quad = \int_C \vec{F} \cdot d\vec{r} = \int_C \langle F_1, F_2, F_3 \rangle \cdot \langle dx, dy, dz \rangle$

scalar form $\quad = \int_C \underbrace{F_1\,dx + F_2\,dy + F_3\,dz}_{\text{"inexact differential"}} = \underbrace{\int_C F_1\,dx + \int_C F_2\,dy + \int_C F_3\,dz}_{\text{single component vector line integrals}}$

Vector Line Integrals (2)

example same semicircle curve as scalar line integral example (see handout) but radius 2. $\boxed{\text{counterclockwise orientation}}$

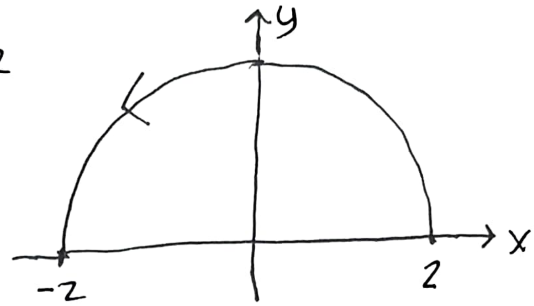
C_1



oppositely directed:

$C_1 = \text{"}-C_2\text{"}$

C_2



$$\vec{r}(t) = \langle \overset{x}{t}, \overset{y}{\sqrt{4-t^2}} \rangle, \quad t=-2..2$$

$$\vec{r}\,'(t) = \langle 1, \frac{-t}{\sqrt{4-t^2}} \rangle$$

$$\vec{r}(\theta) = \langle \overset{x}{2\cos\theta}, \overset{y}{2\sin\theta} \rangle, \quad t = 0..\pi$$

$$\vec{r}\,'(\theta) = \langle -2\sin\theta, 2\cos\theta \rangle$$

vector field:

$$\vec{F} = \langle -y, 2x \rangle = \langle -r\sin\theta, 2r\cos\theta \rangle$$

$$\vec{F}(\vec{r}(\theta)) = \langle -2\sin\theta, 2(2\cos\theta) \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle -\sqrt{4-t^2}, 2t \rangle$$

$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \langle -\sqrt{4-t^2}, 2t \rangle \cdot \langle 1, \frac{-t}{\sqrt{4-t^2}} \rangle$$

$$= -\sqrt{4-t^2} - \frac{2t^2}{\sqrt{4-t^2}} = -\frac{(4-t^2+2t^2)}{\sqrt{4-t^2}}$$

$$= -\frac{(4+t^2)}{\sqrt{4-t^2}} < 0$$

$$\vec{F}(\vec{r}(\theta))\cdot\vec{r}\,'(\theta)$$

$$= \langle -2\sin\theta, 4\cos\theta \rangle \cdot \langle -2\sin\theta, 2\cos\theta \rangle$$

$$= 4\sin^2\theta + 8\underbrace{\cos^2\theta}_{1-\sin^2\theta}$$

$$= 8 - 4\sin^2\theta > 0$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^{\pi} (8-4\sin^2\theta)\, d\theta = 6\pi$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{-2}^{2} -\frac{(4+t^2)}{\sqrt{4-t^2}}\, dt = -6\pi$$

manually insert sign

$$\int_{C_2} \vec{F}\cdot d\vec{r} = -\int_{C_1} \vec{F}\cdot d\vec{r} = -(-6\pi) = 6\pi$$

or

$$= \int_{2}^{-2} -\frac{(4+t^2)}{\sqrt{4-t^2}}\, dt$$

$\uparrow$ $t = 2..-2$ reverses direction

Vector line integrals (3)

$\boxed{\text{3-d example!}}$ twisted cubic $\vec{r}(t) = \langle \overset{x}{t}, \overset{y}{t^2}, \overset{z}{t^3} \rangle$, $t = 0..1$

$\vec{F} = \langle xy, yz, zx \rangle$ (chosen for simple antiderivatives!)

$\vec{F}(\vec{r}(t)) = \langle t(t^2), t^2(t^3), t^3 \cdot t \rangle$ $\quad \vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$= \langle t^3, t^5, t^4 \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle t^3, t^5, t^4 \rangle \cdot \langle 1, 2t, 3t^2 \rangle$

$= t^3(1) + t^5(2t) + t^4(3t^2) = t^3 + 2t^6 + 3t^6$

$= t^3 + 5t^6$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 t^3 + 5t^6 \, dt = \frac{t^4}{4} + \frac{5t^7}{7} \Big|_0^1 = \frac{1}{4} + \frac{5}{7} = \boxed{\frac{27}{28}}$$

<u>same field</u>, straight line from origin to $\langle 1,1,1 \rangle$ (same endpoints)

$\vec{r}_1 = \langle 0,0,0 \rangle$, $\vec{r}_2 = \langle 1,1,1 \rangle$ $\rightarrow$ $\vec{r} = \vec{r}_1 + t(\vec{r}_2 - \vec{r}_1) = t\langle 1,1,1 \rangle$

$t = 0..1$ $\quad = \langle t, t, t \rangle$

$\vec{r}\,'(t) = \langle 1,1,1 \rangle$

$\vec{F}(\vec{r}(t)) = \langle t(t), t(t), t(t) \rangle = \langle t^2, t^2, t^2 \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle t^2, t^2, t^2 \rangle \cdot \langle 1,1,1 \rangle = 3t^2$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 3t^2 dt = \frac{3t^3}{3} \Big|_0^1 = \boxed{1}$$

line integrals of vector fields between two points in general depend on the "path" between them