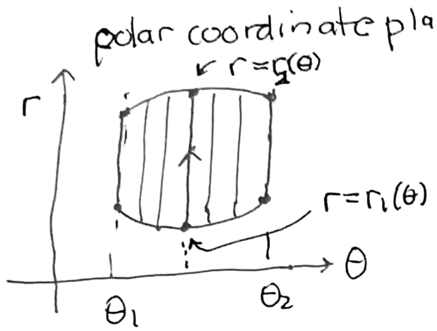
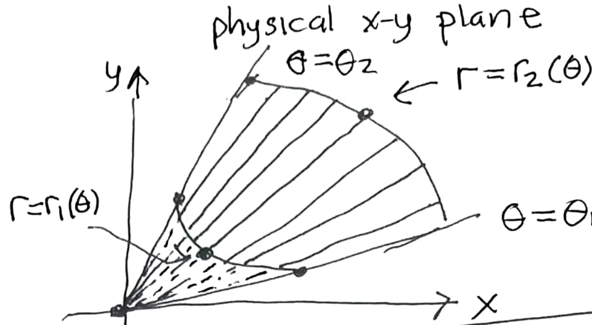
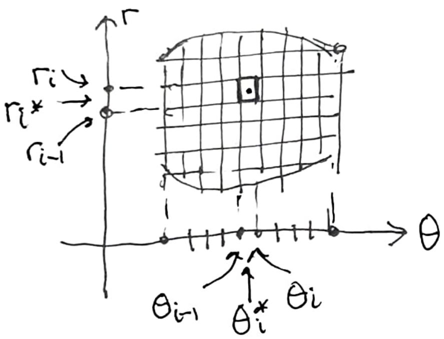
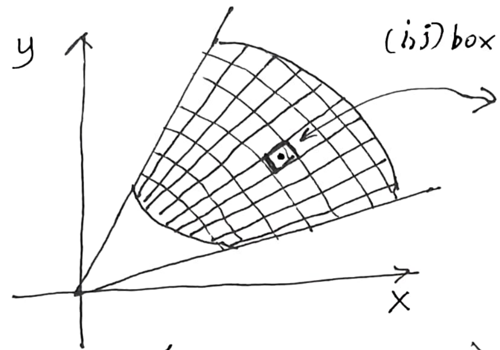
Polar Coordinate Integration (1)



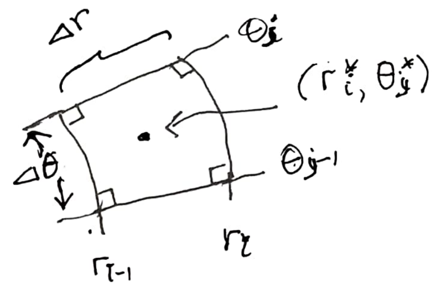


$$\boxed{r = r_1(\theta)\,..\,r_2(\theta) \text{ while } \theta = \theta_1\,..\,\theta_2}$$





region R of plane



$$\Delta A_{i,j} \approx \underbrace{r_i^* \Delta r \Delta\theta}$$

exact if $r_i^* = r_{\text{midpoint}} = \frac{1}{2}(r_{i-1}+r_i)$

$$\approx \underbrace{(\Delta r)}_{\text{arclength}} \underbrace{(r\Delta\theta)}_{\text{arclength}}$$

orthogonal sides

$$\iint_R f\,dA = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} F(r_i^*, \theta_j^*) \underbrace{\Delta A_{ij}}_{r_i^* \Delta r \Delta\theta}$$

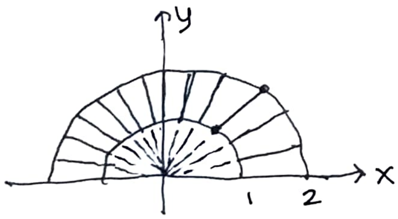
$f(x,y) = f(r\cos\theta, r\sin\theta)$
$\equiv F(r,\theta)$

iterated integral integrand : $F(r,\theta)\,r$

$$= \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} F(r,\theta)\, r\, dr\, d\theta$$

geometric corrective factor : $dA = r\,dr\,d\theta$
$= dr \cdot (r\,d\theta)$

Example (Annulus segment)



$r = 1\,..\,2$ while $\theta = 0\,..\,\pi$

$$\iint \underbrace{3x+4y^2}\, \overbrace{dA}^{r\,dr\,d\theta} = \int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\,dr\,d\theta$$

$3(r\cos\theta) + 4(r\sin\theta)^2$
$= 3r\cos\theta + 4r^2\sin^2\theta$

$$= \dots = \frac{15\pi}{2} \approx 23.562$$

(see Maple worksheet example 4)

15.3 polar coordinate integration

Integration problems initially stated in Cartesian coordinates involving regions better suited to polar coordinates must be converted.

example. Find the volume of the solid region below the paraboloid $\underbrace{z = x^2+y^2}$ inside the cylinder $\underbrace{(x-1)^2+y^2} = 1$.

$z = r^2$

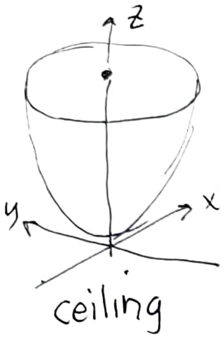
$x^2-2x+1+y^2 = 1$

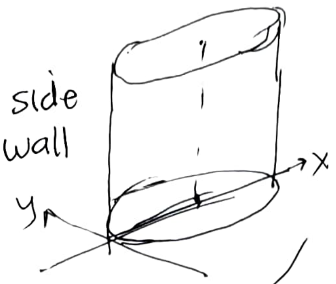
$r^2 - 2(r\cos\theta) = 0$

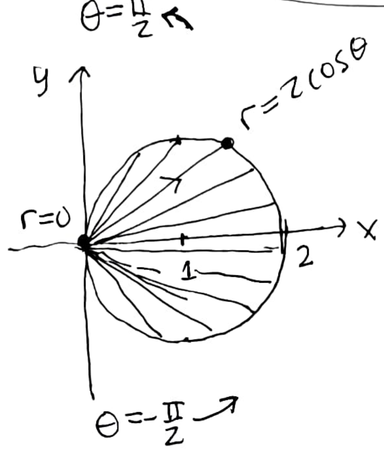
$r(r-2\cos\theta) = 0$

$r = 2\cos\theta$

circle: radius 1, center at $(1,0)$







parametrization in polar coords: $r = 0..2\cos\theta$ while $\theta = -\frac{\pi}{2}..\frac{\pi}{2}$

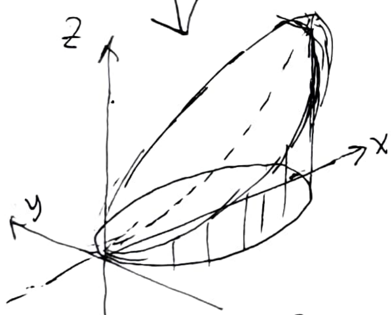
$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} (r^2)\, \underbrace{r\,dr\,d\theta}_{dA}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left. \frac{r^4}{4} \right|_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{16}{4}\cos^4\theta \, d\theta$$

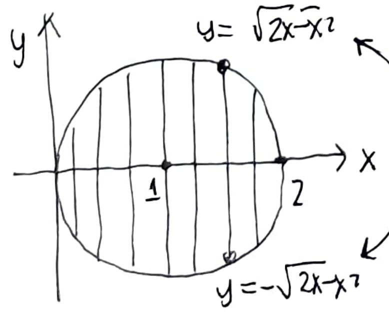
powers of sines, cosines easy for Maple

$$\underset{\text{Maple}}{=} \frac{3\pi}{2} \approx 4.71$$



no need really for this 3d diagram to iterate integral (helps interpret result)

compare:



$x^2+y^2 = 2x$

$y^2 = 2x - x^2$

$y = \pm\sqrt{2x-x^2}$

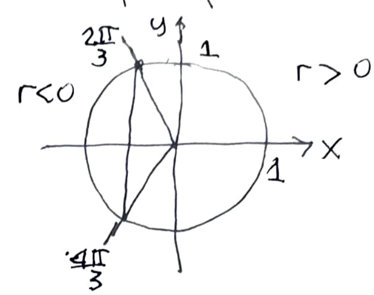
$$\iint_R x^2+y^2\, dA = \int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} x^2+y^2\, dy\, dx$$

hard integral, radicals at outer integral step

Polar Coordinate Integration (3)

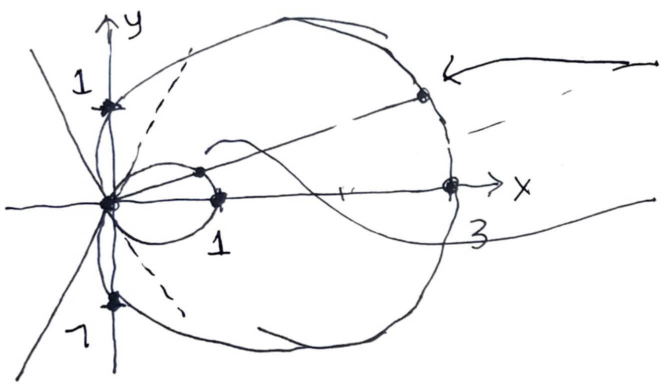
Example. Find area between the inner and outer loops of the cardioid $r = 1 + 2\cos\theta$.

$= 0 \to \cos\theta = -\frac{1}{2}$



$\theta = -\frac{2\pi}{3} \dots \frac{2\pi}{3} \quad r > 0$

$\theta = \frac{2\pi}{3} \dots \frac{4\pi}{3} \quad r < 0$



outer loop

$r = 0 .. \underbrace{1 + 2\cos\theta}_{>0}$ while $\theta = -\frac{2\pi}{3} .. \frac{2\pi}{3}$

inner loop

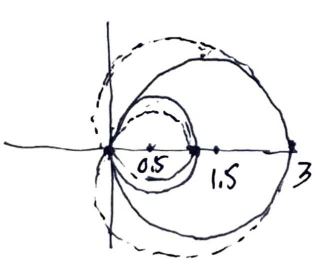
$r = \underbrace{1 + 2\cos\theta}_{<0} .. 0$ while $\theta = \frac{2\pi}{3} .. \frac{4\pi}{3}$

$$A_{outer} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{1+2\cos\theta} 1 \cdot r\,dr\,d\theta = 2\pi + \frac{3\sqrt{3}}{2}$$

$$A_{inner} = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \int_{1+2\cos\theta}^{0} 1 \cdot \underbrace{(-r)}_{>0!}\,dr\,d\theta = \pi - \frac{3\sqrt{3}}{2}$$

↑ ordered, increasing r

$$A_{between} = A_{outer} - A_{inner} = \pi + 3\sqrt{3} \approx 8.338$$



Guesstimate. Circle Center (1.5, 0) radius 1.5 (too small)
center (0.5, 0) radius 0.5 (a bit bigger)

Area $\sim \pi(1.5^2 - 0.5^2) \approx 6.28$

in the right ball park!