14.5b chain rule etc

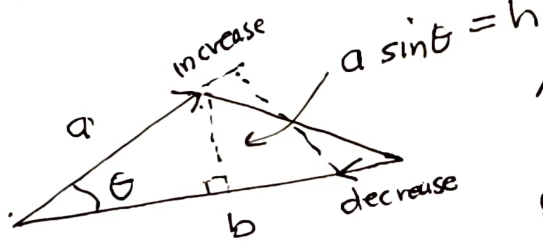


The authors spend many pages on tree diagrams for partial derivatives which they seem to think as the greatest thing since sliced bread but I dont buy it. Form your own opinion.

YET not a word on how the chain rule applies to **related rates problems** from calc I and there are a number of homework problems on this.

Given a set of variables which satisfy a constraint, the derivative of that constraint is a constraint on their rates of change. Given the values of all the variables and of all their rates of change EXCEPT ONE, one can solve that derivative constraint for the remaining rate of change

Example




A triangle has constant area

$A = \frac{1}{2} a b \sin\theta$

$\frac{da}{dt} = 2 \text{ cm/s}, \quad \frac{db}{dt} = -3 \text{ cm/s}$

If initially $a = 20$ cm, $b = 30$ cm what is the rate of change of the included angle $\theta$ which has the initial value $\pi/6$?

The initial area is $A = \frac{1}{2}(20)(30)\sin\frac{\pi}{6} = 300\left(\frac{1}{2}\right) = 150 \quad (\text{cm}^2)$

Differentiate the constraint: $\frac{d}{dt}\left[\frac{1}{2} a b \sin\theta = 150\right]$

$$\frac{1}{2}\left(\frac{da}{dt} b + a\frac{db}{dt}\right)\sin\theta + \frac{1}{2} a b \cos\theta \underbrace{\frac{d\theta}{dt}}_{\text{isolate}} = 0$$

$$\frac{d\theta}{dt} = -\frac{\sin\theta}{ab\cos\theta}\left(\frac{da}{dt} b + a\frac{db}{dt}\right) = -\tan\theta\left(\frac{1}{a}\frac{da}{dt} + \frac{1}{b}\frac{db}{dt}\right)$$

at initial time:

$$\left.\frac{d\theta}{dt}\right|_0 = -\underbrace{\tan\frac{\pi}{6}}_{1/\sqrt{3}}\left(\frac{1}{20}(3) + \frac{1}{30}(-2)\right) = -\frac{1}{10\sqrt{3}}\underbrace{\left(\frac{3}{2} - \frac{2}{3}\right)}_{5/6} = \frac{1}{2\sqrt{3}\cdot 6}$$

$$= -\frac{1}{12\sqrt{3}} \text{ rad/sec} \approx -4.2\ {}^\circ/\text{s} \quad \text{decreasing}$$

$\boxed{\text{Chain rule?}}$

$$0 = \frac{d}{dt}A = \frac{\partial A}{\partial a}\frac{da}{dt} + \frac{\partial A}{\partial b}\frac{db}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -\left(\frac{\partial A}{\partial a}\frac{da}{dt} + \frac{\partial A}{\partial b}\frac{db}{dt}\right) \Big/ \frac{\partial A}{\partial \theta}$$

now we have names for the coefficients of the derivatives here but we already know how to do this

**Example** Ideal Gas Law games

$$F(P,V,T) = PV - nRT = 0$$

we can repeat this for a non ideal gas using implicit differentiation.

cyclic product: $\underbrace{\frac{\partial P}{\partial V}\Big|_T}\ \underbrace{\frac{\partial V}{\partial T}\Big|_P}\ \underbrace{\frac{\partial T}{\partial P}\Big|_V} = -1$ is a consequence of the chain rule

we can think of each of these variables as an implicit function of the other two variables, so we can take 2 partial derivatives for each of these variable arrangements, holding the other independent variable fixed in each case.

Why?

$F(x,y,z) = 0$ can determine each variable as a function of the other two

$\rightarrow z = z(x,y) \mapsto F(x,y,z(x,y)) = 0$ differentiate

$$0 = \frac{\partial}{\partial x} F(x,y,z(x,y)) = \frac{\partial F}{\partial x}(x,y,z(x,y)) + \frac{\partial F}{\partial z}(x,y,z(x,y))\frac{\partial z}{\partial x}\Big|_y$$

solve: $\frac{\partial z}{\partial x}\Big|_y = -\frac{\partial F/\partial x}{\partial F/\partial z}$

$\rightarrow x = x(y,z)$ repeat... $\frac{\partial x}{\partial y}\Big|_z = -\frac{\partial F/\partial y}{\partial F/\partial x}$ ← cyclicly change $(x,y,z)$

$\rightarrow y = y(x,z)$ repeat... $\frac{\partial y}{\partial z}\Big|_x = -\frac{\partial F/\partial z}{\partial F/\partial y}$ ditto

$$\frac{\partial z}{\partial x}\Big|_y \frac{\partial x}{\partial y}\Big|_z \frac{\partial y}{\partial z}\Big|_x = \left(-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\right)\left(-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}\right)\left(-\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}}\right) = (-1)^3 = -1 \checkmark$$

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$\boxed{\text{cute but so what?}}$

in above problem get third physical rate of change for free!

$\boxed{14.5b}$ chain rule etc



There are no WebAssign problems on converting second derivatives! So here is an example. WARNING: it is a bit tedious so we have to be organized.

$\boxed{14.5.50}$ If $u = f(x,y)$ and $x = e^s \cos t,\ y = e^s \sin t$

show that $\underbrace{u_{xx} + u_{yy}}_{\text{"Laplacian"}} = e^{-2s}(u_{ss} + u_{tt})$

$$x = e^s \cos t \quad \frac{\partial x}{\partial s} = e^s \cos t \quad \frac{\partial x}{\partial t} = -e^s \sin t$$

$$y = e^s \sin t \quad \frac{\partial y}{\partial s} = e^s \sin t \quad \frac{\partial y}{\partial t} = e^s \cos t$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = e^s \cos t\, u_x + e^s \sin t\, u_y \qquad u \to u_x \qquad u \to u_y$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = -e^s \sin t\, u_x + e^s \cos t\, u_y$$

$$\frac{\partial^2 u}{\partial s^2} = e^s \cos t\, u_x + e^s \cos t\, \frac{\partial u_x}{\partial s} + e^s \sin t\, u_y + e^s \sin t\, \frac{\partial u_y}{\partial s} \quad \text{(product rule)}$$

$$e^s \cos t\,(u_x)_x + e^s \sin t\,(u_x)_y \qquad e^s \cos t\,(u_y)_x + e^s \sin t\,(u_y)_y$$

$$= e^s(\cos t\, u_x + \sin t\, u_y)$$
$$+ e^{2s}[\underbrace{\cos^2 t\, u_{xx}} + \cos t \sin t\, u_{xy} + \sin t \cos t\, u_{yx} + \underbrace{\sin^2 t\, u_{yy}}]$$

cancel

$$\frac{\partial^2 u}{\partial t^2} = -e^s \cos t\, u_x - e^s \sin t\, \frac{\partial u_x}{\partial t} - e^s \sin t\, u_y + e^s \cos t\, \frac{\partial u_y}{\partial t} \quad \text{(product rule)}$$

$$-e^s \sin t\,(u_x)_x + e^s \cos t\,(u_x)_y \qquad -e^s \sin t\,(u_y)_x + e^s \cos t\,(u_y)_y$$

$$= e^s(-\cos t\, u_x - \sin t\, u_y)$$
$$+ e^{2s}[\underbrace{+\sin^2 t\, u_{xx}} - \sin t \cos t\, u_{xy} - \cos t \sin t\, u_{yx} + \underbrace{\cos^2 t\, u_{yy}}]$$

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = e^{2s}\left(\underbrace{(\cos^2 t + \sin^2 t)u_{xx}} + 0 + \underbrace{(\sin^2 t + \cos^2 t)\, u_{yy}}\right)$$
$$= e^{2s}(u_{xx} + u_{yy})$$

$$\hookrightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\underbrace{\left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right)}_{\text{Laplacian in variables } (s,t)}$$

This is my third attempt and you can still see corrections above. Maple makes this easy.