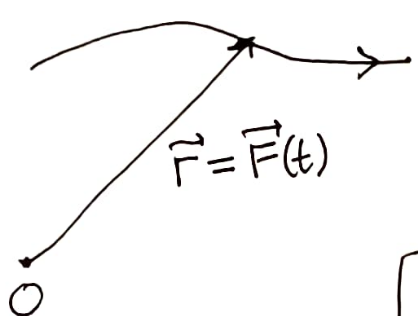
$\boxed{13.1}$ vector calculus = calculus of spacecurves

We begin $\boxed{\text{multivariable calculus}}$ by considering multiple dependent variables which are functions of a single independent variable. We package the dependent variables as a vector and apply single variable calculus operations simultaneously to each component.

$\boxed{\text{Parametrized curves}}$ in $\mathbb{R}^n$ can be understood through the lens of $n=3$, space. We interpret them as position vectors in space as a function of a parameter t.

$\boxed{\text{vector-valued function}}$: $\vec{F}(t) = \langle F_1(t), F_2(t), F_3(t) \rangle$

visualized as a parametrized curve in space:



$$\vec{r} = \vec{F}(t)$$

equivalent to

$$\langle x, y, z \rangle = \langle F_1(t), F_2(t), F_3(t) \rangle$$

but usually we have no named function and simply write.

$$\boxed{\begin{array}{l} \vec{r} = \langle x(t), y(t), z(t) \rangle \\ \text{or sometimes } \vec{F}(t) = \langle x(t), y(t), z(t) \rangle \end{array}}$$

where these are explicit expressions

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example: $\boxed{\text{twisted cubic}}$

$$\vec{r} = \langle t, t^2, t^3 \rangle, \quad -1 \le t \le 1$$

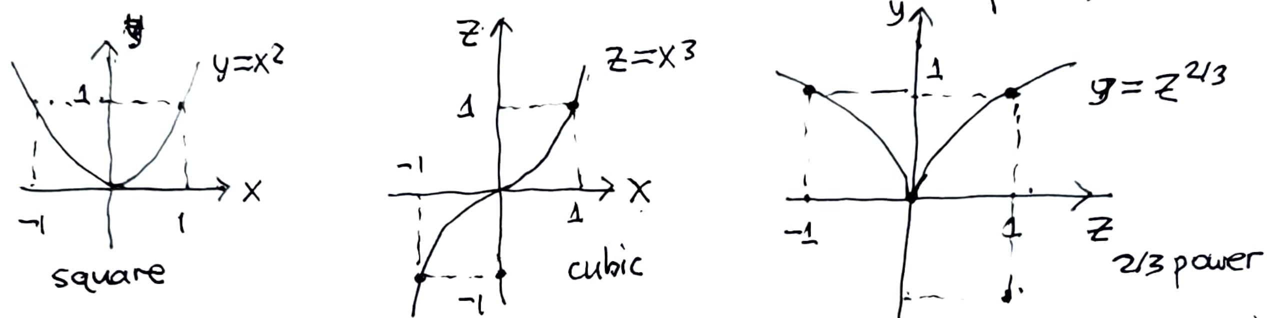
or $x = t,\ y = t^2,\ z = t^3$

eliminate parameter

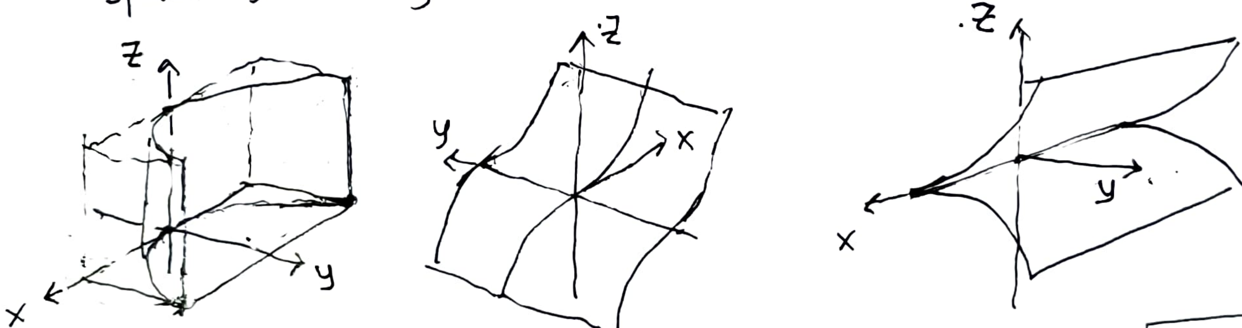
$y = x^2$ $\quad z = x^3$

$y = (z^{1/3})^2 = z^{2/3}$

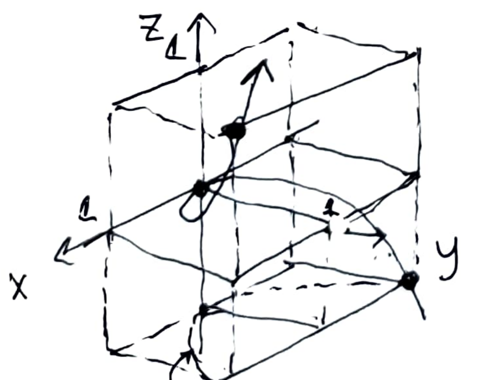
these happen to be the projections of the curve onto the 3 coordinate planes



Each is independent of the third coordinate, so represent surfaces in space by extending them in that third direction:



Intersection of any two of these surfaces describes the $\boxed{\text{unparametrized curve}}$



$y = x^2$ imagine projected up to intersect actual curve

endpoints: $\langle -1, 1, -1 \rangle$
$\langle 1, 1, 1 \rangle$
$t = 0$: $\langle 0, 0, 0 \rangle$
} 3 key points in diagram

This is the job of technology

[see Maple]

increase t, move up (z monotonically increasing)

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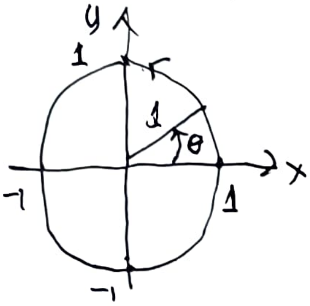
Reverse process: parametrizing a curve specified as the intersection of two surfaces

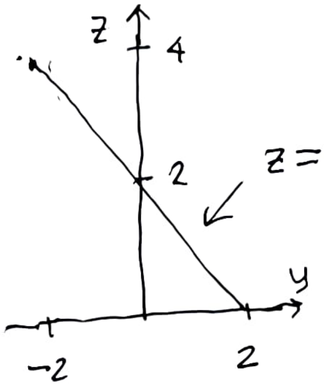
example: plane cut cylinder

$$\boxed{x^2+y^2=1\ ,\ y+z=2}$$

how to parametrize?

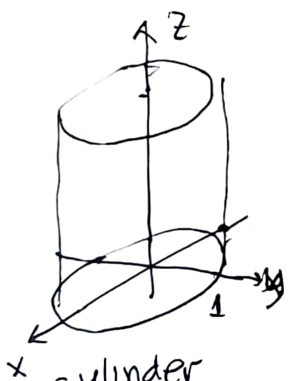


unit circle
choose $t=\theta$:
$x=\cos t\ ,\ y=\sin t$
$t=0..2\pi$



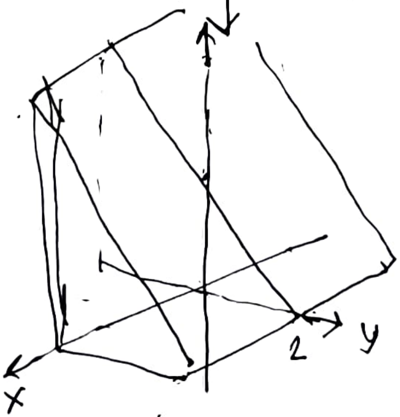
$z=2-y \rightarrow z=2-\sin t$

$$\boxed{\begin{gathered}\langle x,y,z\rangle=\langle\cos t,\sin t,2-\sin t\rangle\\ t=0..2\pi\end{gathered}}$$



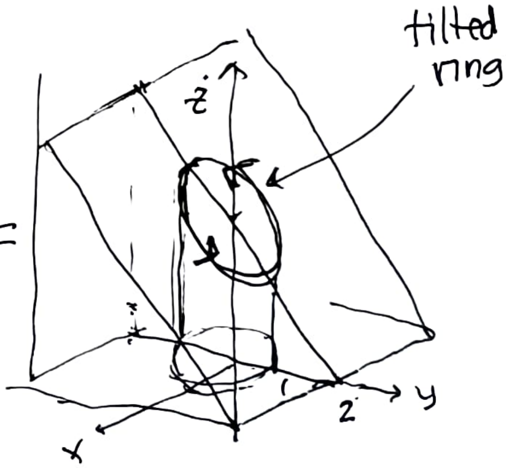
cylinder

+



plane

=



oblique cut of cylinder

increase t, move in counterclockwise direction from above.

[see Maple]

vector calculus = calculus of spacecurves

Calculus operations (lim, diff, int) on a vector-valued function are performed simultaneously on all components.

"parallel computation"

$$\square\, \vec{r}(t) = \langle \square\, x(t), \square\, y(t), \square\, z(t) \rangle$$

↑ put calc1 operation in the box.

example $\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$ what is the domain?

$3-t>0$, $3>t$, $t<3$; $t \geq 0$ } $0 \leq t < 3$

only need limits at endpoints (continuous inside interval)

$$\lim_{t\to 0^+} \vec{r}(t) = \langle 0^3, \ln(3-0), \sqrt{0} \rangle = \langle 0, \ln 3, 0 \rangle$$

one-sided limit

$$\lim_{t\to 3} \vec{r}(t) = \langle 3^3, \underbrace{\lim_{t\to 3} 3-t}_{=-\infty}, \underbrace{\lim_{t\to 3} \sqrt{t}}_{\sqrt{3}} \rangle = \langle 27, -\infty, \sqrt{3} \rangle$$

limit does not exist

but does approach $x=27, z=\sqrt{3}$ in x-z planeprojection while $y \to -\infty$, so useful information

$$\vec{r}\,'(t) = \frac{d}{dt}\vec{r}(t) = \langle \frac{d}{dt} t^3, \frac{d}{dt} \ln(3-t), \frac{d}{dt} t^{1/2} \rangle$$

$$= \langle 3t^2, \frac{1}{3-t}(-1), \frac{1}{2} t^{-1/2} \rangle = \langle 3t^2, \frac{1}{t-3}, \frac{1}{2\sqrt{t}} \rangle$$

$$\int \vec{r}(t)\,dt = \langle \int t^3\,dt, \int \ln(3-t)\,dt, \int t^{1/2}\,dt \rangle$$

$$= \langle \frac{t^4}{4} + C_1, ? + C_2 \text{ (use Maple)}, \frac{t^{3/2}}{3/2} + C_3 \rangle$$

(integration by parts — NO!)