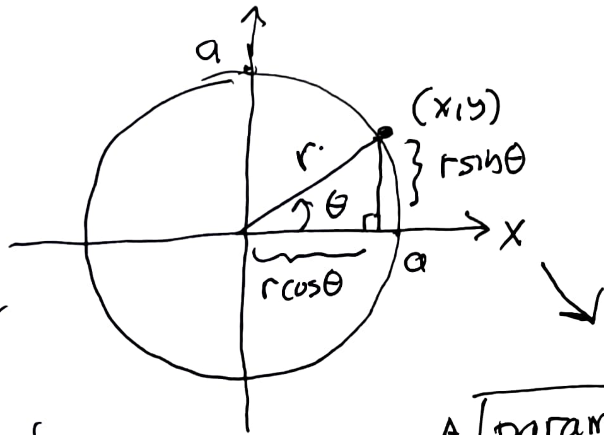
10.1 parametrized curves in the plane (calc 2 review detour)

Any single condition in (x,y) determines a curve in the plane consisting of all its solution points. On the other hand to describe the motion of a point in the plane (physics!), moving around as a function of time, we need parametrized curves.

Example



A $\boxed{\text{curve}}$ is a set of points along a path.

$x^2+y^2=a^2$ (Cartesian coords)

or

$r=a$ (polar coords)

The soln set is a circle at the origin

A $\boxed{\text{parametrized curve}}$ is a way of tracing out a path.

$x = a\cos\theta$

$y = a\sin\theta$

↑ dependent variables

↑ independent var. "parameter"

choose $0 \le \theta \le 2\pi$ to trace out the circle starting at $(a,0)$ and going around once counterclockwise.

The default variable name for parametrized curve parameters in calc is "t" (often for time)

$$\boxed{x = a\cos t,\ y = a\sin t,\ 0 \le t \le 2\pi}$$

Indeed we can animate curves by correlating t with actual time to visualize tracing out the curve as the parameter increases "uniformly"!

Eliminating the parameter we can go backwards:

$\frac{x}{a}=\cos t,\ \frac{y}{a}=\sin t \rightarrow 1=\cos^2 t+\sin^2 t=\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2=\frac{x^2+y^2}{a^2} \rightarrow x^2+y^2=a^2$

In this case we recognize the circle and its location and radius from the corresponding $\boxed{\text{unparametrized curve}}$

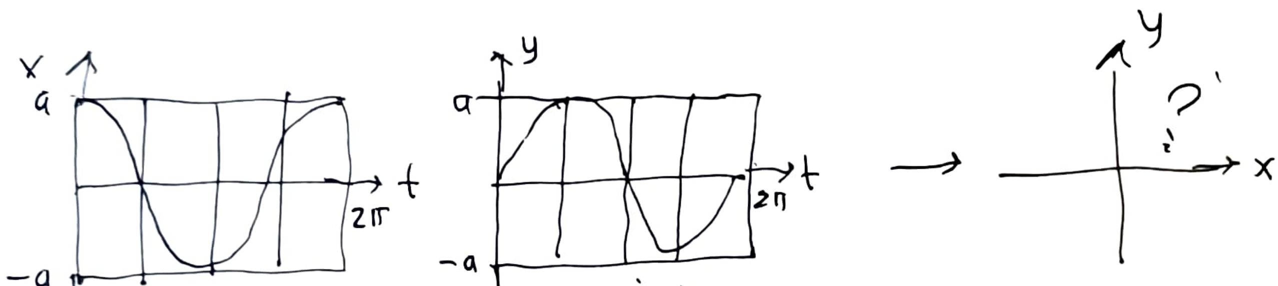
changing the parametrization changes how we trace out the curve.

$x = a\cos t,\ y = a\cos t \xrightarrow{\text{let } t \to -t}$ $x = a\cos(-t) = a\cos t$
$y = a\sin(-t) = -a\sin t$
choose $0 \le t \le 2\pi$

now we start at $(a,0)$ and move one revolution clockwise around the circle.

A parametrization "orients" a curve giving it a forward direction (increasing t) at each point as it is being traced out.

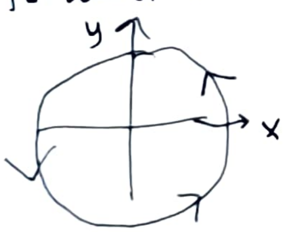
we need technology to help us visualize curves in general.



$x = a\cos t$ $y = a\sin t$

not easy to see how x & y versus t translate to y versus x!

A $\boxed{\text{directed curve}}$ has a unique forward direction at each point

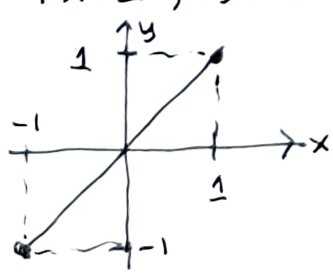


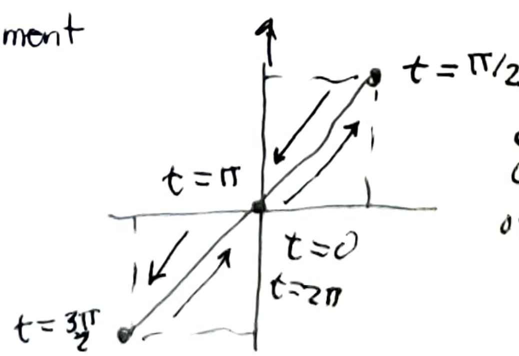
but a parametrized curve can change direction if it retraces the same pts.

$\underline{\text{example}}$ $x = \sin t,\ y = \sin t,\ t = \underbrace{0..2\pi}$

Maple notation: $\underset{\text{2 dots}}{a..b}$ (start: a, finish: b)

eliminate t: $y = x$
but $|x| \le 1,\ |y| \le 1$ so line segment





one complete oscillation

$\boxed{10.1}$ parametrized curves in the plane

example $\begin{cases} x = t^2 - 2t = t(t-2) \\ y = t+1 \\ t = 0..4 \end{cases}$

key points to label in graph:

y-intercepts: $x = 0 = t(t-2) \rightarrow t = 0, 2 \rightarrow y = 1, 3 : (0,1), (0,3)$

x-intercepts: $y = 0 = t+1 \rightarrow \underbrace{t = -1}_{\text{outside parameter range}} \rightarrow x = 1 - 2(-1) = 3$

endpoints: $t=0$: $x = 0, y = 1$: $(0,1)$

$t=4$: $x = 16 - 2(4) = 8$: $(8,5)$

$y = 4+1 = 5$

eliminate parameter:

$y = t+1 \rightarrow t = y-1 \rightarrow x = (y-1)^2 - 2(y-1) = y^2 - 2y + 1 - 2y + 2$

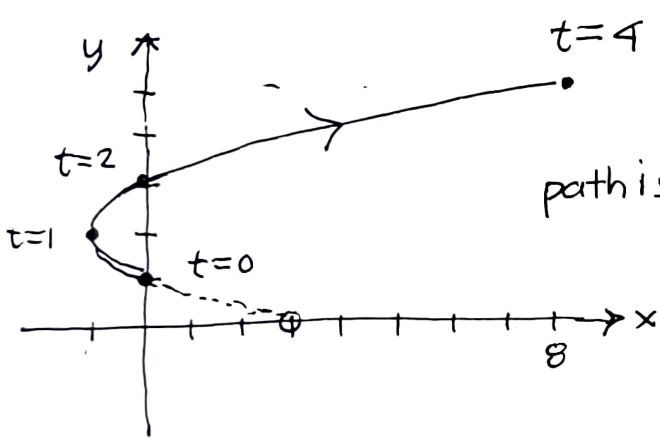
$= y^2 - 4y + 3$ parabola! on its side

$= (y-2)^2 - 4 + 3$

$= (y-2)^2 - 1$

minimum value at $y = 2$ (vertex) : $x = -1$

$(-1, 2)$ $t = 1$



path is a directed curve segment

example $x = t - \sin t$, $y = 1 - \cos t$, $t = 0..\infty$

cannot eliminate parameter

[see Maple]