Partial Integration

function of multiple variables $f(x,y,\dots)$

partial differentiation with respect to any one of these variables means treat all the other variables as constants and apply the rules of differentiation w.r.t. the chosen variable.

partial integration with respect to any one of these variables means treat all the other variables as constants and integrate with respect to the chosen variable

indefinite partial integration: find an anti-derivative, add an arbitrary constant

definite partial integration: subtract the values of the antiderivative at the upper & lower limits

The order of two successive such operations does not matter

$$\frac{\partial^2 f}{\partial x \partial y}(x,y,\dots) = \frac{\partial^2 f}{\partial y \partial x}(x,y,\dots) \quad \leftarrow \text{iterated Calc I derivatives}$$

$$\int_{y_1}^{y_2}\left(\int_{x_1}^{x_2} f(x,y,\dots)\cdot dx\right) dy = \int_{x_1}^{x_2}\left(\int_{y_1}^{y_2} f(x,y,\dots)\, dy\right) dx$$

These are called "iterated" integrals, just successive Calc II operations you know how to do

The calc III aspect is understanding the meaning of these iterated integrals.

$\boxed{\text{Evaluating them is not the problem.}}$

partial derivatives / integrals "commute"

$$f(x,y) = xy^2$$

$$f_x(x,y) = \frac{\partial}{\partial x}(xy^2) = y^2 \longrightarrow f_{xy}(x,y) = \frac{\partial}{\partial y}(y^2) = 2y$$

$$\| \ ?$$

$$f_y(x,y) = \frac{\partial}{\partial y}(xy^2) = 2xy \rightarrow f_{yx}(x,y) = \frac{\partial}{\partial x}(2xy) = 2y$$

$$\therefore \quad \frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$$

nested integrals (double integrals)

$$\int_a^b f(x,y)\,dx = \int_a^b xy^2\,dx = \left.\frac{x^2y^2}{2}\right|_{x=a}^{x=b} = (b^2-a^2)\frac{y^2}{2}$$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \frac{b^2-a^2}{2} y^2\,dy = \left.\frac{b^2-a^2}{2}\,\frac{y^3}{3}\right|_{y=c}^{y=d}$$

$$= \left(\frac{b^2-a^2}{2}\right)\left(\frac{d^3-c^3}{3}\right)$$

$$\int_c^d f(x,y)\,dy = \int_c^d xy^2\,dy = \left.\frac{xy^3}{3}\right|_{y=c}^{y=d} = \frac{x}{3}(d^3-c^3)$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \frac{x}{3}(d^3-c^3)\,dx = \left.\frac{d^3-c^3}{3}\,\frac{x^2}{2}\right|_{x=a}^{x=b}$$

$$= \left(\frac{d^3-c^3}{3}\right)\left(\frac{b^2-a^2}{2}\right)$$

equal !

$\boxed{\text{evaluating}}$ double and triple integrals (definite integrals) you already know how to do — it is just a matter of successive Calc II integrations with respect to distinct variables.

These are "nested" integrals like "nested" parentheses.

$$(\,(\ \ldots\ )\,) \qquad\qquad \int \int \ \ldots\ dy\ dx$$

must match in pairs — ditto

← now more than ever these differentials MATTER!