"u-substitution"

$$H = \int_a^b f(x)\,dx = \int_{x=a}^{x=b} f(x)\,dx$$

identify in integrand an expression $u = u(x)$ to be used as the new independent variable

Then:

$$u = u(x) \xrightarrow[\text{for } x]{\text{solve}} x = x(u)$$

$$du = u'(x)\,dx \longrightarrow dx = \frac{du}{u'(x)} = \frac{du}{u'(x(u))}$$

replace x by $x(u)$

now re-express integrand and differential by replacing x and dx there using this transformation.

simplify and find antiderivative wrt u if possible.

definite integral: $x = a \rightarrow u = u(a) = A$
$x = b \rightarrow u = u(b) = B$
change limits

$$H = \int_{u=A}^{u=B} f(x(u))\,dx(u) = \int_A^B f(x(u))\,dx(u)$$

$$= F(u)\Big|_A^B = \ldots$$

indefinite integral:

$$\int f(x)\,dx = F(u) + C_1 = F(u(x)) + C_1$$

revert to original variable

example:

$$H = \int_4^9 \frac{x}{\sqrt{x}-1}\,dx = \int_{x=4}^{x=9} \frac{x \rightarrow 2(u+1)^2}{\sqrt{x}-1 \rightarrow u}\, dx \rightarrow 2(u+1)\,du$$

$$u = \sqrt{x} - 1 = x^{1/2} - 1 \longrightarrow x^{1/2} = u + 1 \rightarrow x = (u+1)^2$$

$$du = \frac{1}{2}x^{-1/2}\,dx = \frac{dx}{2x^{1/2}} \longrightarrow dx = 2x^{1/2}\,du = 2(u+1)\,du$$

$x = 4$: $u = \sqrt{4} - 1 = 2 - 1 = 1$
$x = 9$: $u = \sqrt{9} - 1 = 3 - 1 = 2$

$$H = \int_{u=1}^{u=2} \frac{(u+1)^2}{u}\left(2(u+1)\,du\right)$$

$$= 2\int_1^2 \frac{(u+1)^3}{u}\,du = 2\int_1^2 \frac{u^3+3u^2+3u+1}{u}\,du$$

$$= 2\int_1^2 u^2 + 3u + 3 + \frac{1}{u}\,du$$

$$= 2\left(\frac{u^3}{3} + \frac{3}{2}u^2 + 3u + \ln u\right)\Big|_1^2$$

$$= \ldots = \frac{59}{3} + 2\ln 2$$

indefinite integral:

$$\int \frac{x}{\sqrt{x}-1}\,dx = 2\left(\frac{(\sqrt{x}-1)^3}{3} + \frac{3}{2}(\sqrt{x}-1)^2 + 3(\sqrt{x}-1) + \ln(\sqrt{x}-1)\right) + C_1$$

$$\overset{\text{expand out}}{=} \underbrace{\frac{2}{3}x^{3/2} + x + 2x^{1/2} + \ln(\sqrt{x}-1)}_{\text{Maple result.}} \underbrace{- \frac{11}{3} + C_1}_{= C}$$

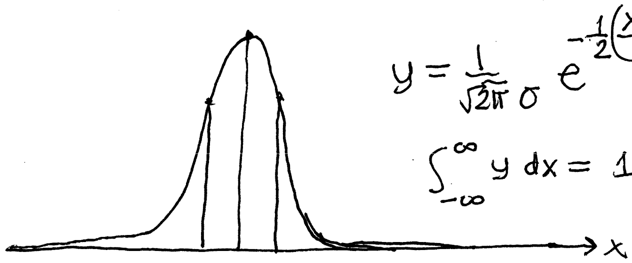
antiderivatives can differ by a constant!

alternatively you can return to the original variable for the final evaluation:

$$H = 2\left(\frac{(\sqrt{x}-1)^3}{3} + \frac{3}{2}(\sqrt{x}-1)^2 + 3(\sqrt{x}-1) + \ln(\sqrt{x}-1)\right)\Big|_4^9$$

$$= \ldots$$

the bell curve (normal distribution): an important example of definite integral change of variable



$\mu-\sigma \quad \mu \quad \mu+\sigma$

symmetric about peak

$$y = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\int_{-\infty}^{\infty} y\, dx = 1$$

The bell curve describes the distribution of values of many physical variables, like the height of adult males in a given population — there is a peak around some average values, some spread about that peak and then it falls off rapidly away from that peak.

If we move the peak to the origin and measure the variable x in units of the so called standard deviation σ, we get a standard bell curve. Areas under a general bell curve correspond directly to areas under the standard bell curve, and so probability questions can instead be answered by referring them to that standard curve: $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}$, $\quad u = \frac{x-\mu}{\sigma} \quad du = \frac{dx}{\sigma}$

The probability that x takes a value in the interval $a \le x \le b$ is

$$P(a \le x \le b) = \int_a^b \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{dx}{\sigma} = \int_{x=a}^{x=b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$= \int_{u_a}^{u_b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du, \quad \text{where} \quad u_a = \frac{a-\mu}{\sigma},\ u_b = \frac{b-\mu}{\sigma}$$

In the old days the area function for this function graph was tabulated so that probability questions could be answered — since technology did not exist to calculate these areas.

Now we have the special function "$\frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$" ("error function") defined to be the area function zeroed at the origin, so that

$$P(u_a \le u \le u_b) = \int_{u_a}^{u_b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = \frac{1}{2}\operatorname{erf}\left(\frac{u_b}{\sqrt{2}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{u_a}{\sqrt{2}}\right),$$

The factors of 2 require an explanation for another day.
[Another change of variable: $\frac{1}{\sqrt{2}} e^{-\frac{1}{2}u^2} du = e^{-v^2} dv$ if $v = u/\sqrt{2}$ goggle it]