Definite integrals approximated with a power series

$$\int_0^{0.3} \underbrace{\frac{x^5}{1+x^7}} \, dx$$

expand in series, then integrate term by term to find an antiderivative plus C, set C=0 then evaluate antiderivative at upper/lower limits of integration to evaluate the definite integral

geometric series trick:

$$\frac{x^5}{1+x^7} = x^5 \cdot \frac{1}{1-(-x^7)} = x^5 \sum_{n=0}^{\infty} (-x^7)^n = \sum_{n=0}^{\infty} (-1)^n x^{7n} \cdot x^5 = \sum_{n=0}^{\infty} (-1)^n x^{7n+5}$$

$$= x^5 - x^{12} + x^{19} - x^{26} + \dots$$

(alternating series!)

integrate term by term:

$$\int \frac{x^5}{1+x^7} dx = \underbrace{\frac{x^6}{6} - \frac{x^{13}}{13} + \frac{x^{20}}{20} - \frac{x^{27}}{27} + \dots}_{} + C = \int \left(\sum_{n=0}^{\infty} (-1)^n x^{7n+5} \right) dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^{7n+5} dx$$

$$= \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+6}}{7n+6}} + C$$

choose antiderivative equal to zero at $x=0$ (set $C=0$)

$$\int_0^{0.3} \frac{x^5}{1+x^7} dx = \left[\frac{x^6}{6} - \frac{x^{13}}{13} + \frac{x^{20}}{20} - \frac{x^{27}}{27} + \dots \right] \Big|_0^{0.3} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+6}}{7n+6} \Big|_0^{0.3}$$

$$= \frac{0.3^6}{6} - \frac{0.3^{13}}{13} + \frac{0.3^{20}}{20} - \frac{0.3^{27}}{27} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{(0.3)^{7n+6}}{7n+6}$$

$- 0$ (no contribution from lower limit)

$$= 0.\underbrace{000121}50 - \underbrace{1.2 \times 10^{-8} + 1.7 \cdot 10^{-12} - 2.8 \cdot 10^{-16} + \dots}$$

6 decimal places

first term rounds up to 0.000122 but

total error in rest of series $< 1.2 \times 10^{-8}$ by alternating series estimate so shouldn't matter, but second term affects roundoff in 6th digit!

$$\underbrace{0.00012148}$$

first 2 terms changes final digit when rounding

$$\approx \boxed{0.000121}$$

truncating series after first term gives accuracy to 6 decimal places but rare roundoff situation changes final digit