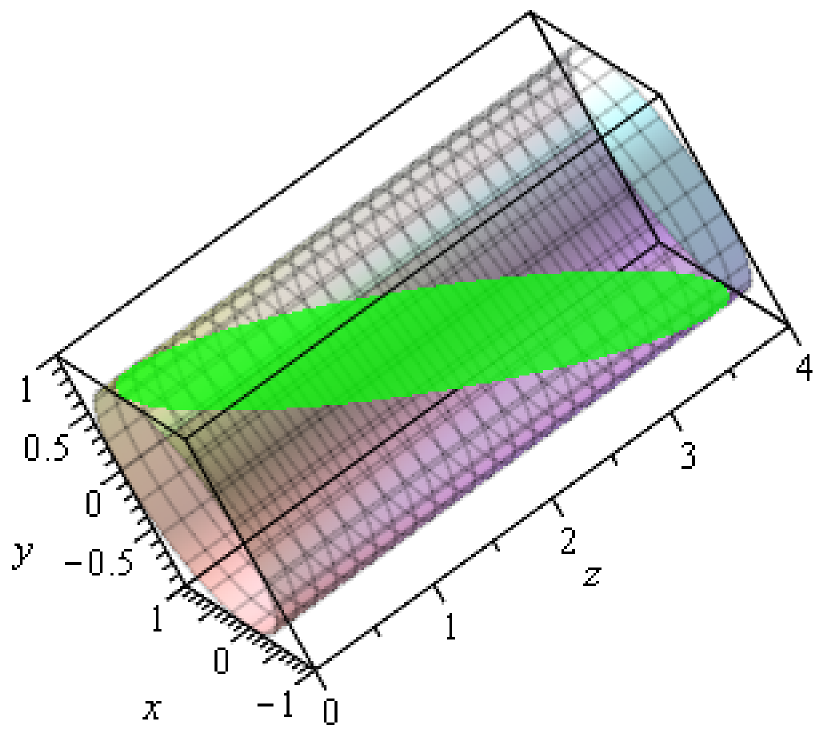
Stewart Calculus 8e 6Plus.4a. (on steroids)

Fall 2015 bob jantzen villanova university

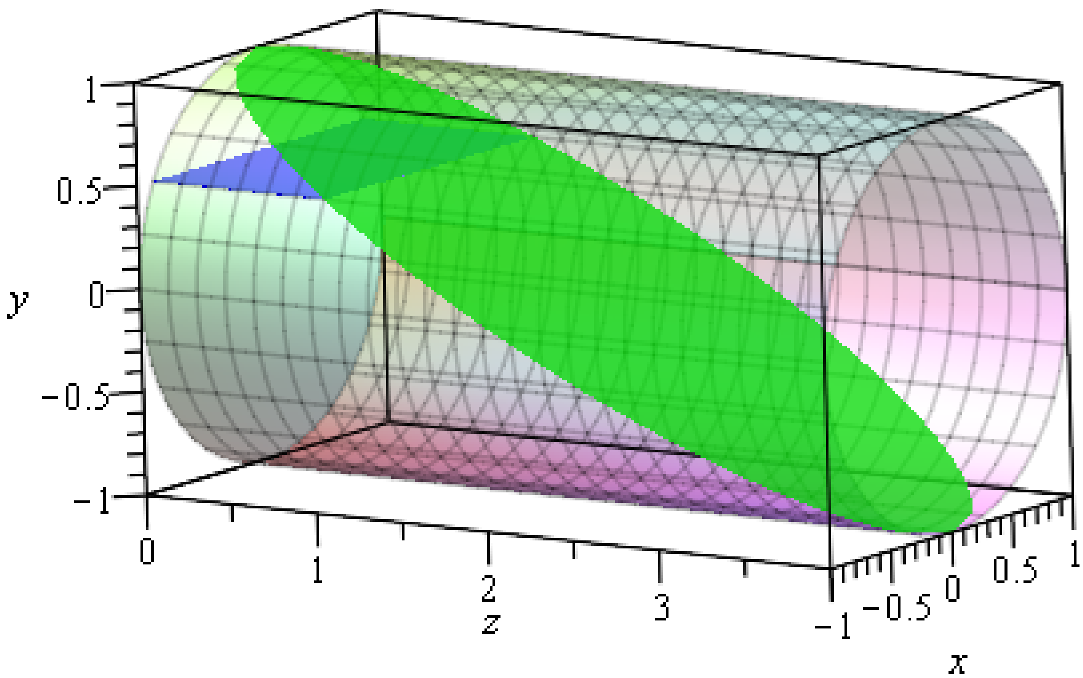
A cylindrical glass of radius *r* and height *L* is filled with water and then tilted until the water remaining in the glass exactly covers its base.



Determine a way to "slice" up the water into parallel rectangular cross-sections and then set up a definite integral for the volume of the water in the glass.

hint

We rotate it so the axis of symmetry of the cylinder is horizontal to analyze this in terms of the plane cross-sections which are parallel to that axis, setting up our *x*-*y*-*z* axes aligned with the cylinder. The horizontal blue rectangle in this view is a typical such cross-section.



Set up your x-y axes on one face (bottom) of the cylinder and let the z axis be along the symmetry axis of the cylinder. Obtain the area $A(y)$ of the horizontal cross-section rectangles and integrate it appropriately to get the volume of the half of the cylinder containing the water, which by symmetry is half the volume of a cylinder with the given radius and length: $V_{cylinder} = \pi r^2 L$, so you know what the result should be $V_{water} = \frac{1}{2}\pi r^2 L$. The key question here is can you parametrize the lengths of the two sides of the rectangular cross-sections in terms of the single variable y marking the left end face level of those cross-sections?

▶ plot and solution

▼ dr bob's water pumping work problem: pump the water to the rim,

how much work?

Now tilt this diagram back up so the green upper surface is horizontal, the water level in a cylindrical tank open at the right face.

a) If $y = Y$ is the level at the left end face of the cylindrical tank of a plane cross-section E_Y of this cylindrical tank parallel to the green face, where $-r \le Y \le r$, set up an integral giving the area of this cross-section $A(Y)$ and evaluate it to get the known value of the volume of this diagonal half cylinder: $V = \int_{-r}^{r} A(Y)\, \mathrm{d}h(Y)$, where $h(Y)$ (proportional to Y) is the height of the green surface above the cross-section E_Y.

Note that to evaluate $A(Y)$ as a iterated integral with respect to x, one needs to introduce a proportional variable $z(x)$ which measures length along the direction parallel to the green face, since area results only from integrating a length of a line segment cross-section with respect to a variable measuring length in the perpendicular direction, as is the case for integrating a cross-sectional area to get volume.

b) Now evaluate the work ratio $\frac{W}{\rho g} = \int_{-r}^{r} A(Y)\, h(Y)\, \mathrm{d}h(Y)$, where W is the work done to pump water up to the bottom of the right end face rim where it can flow out, and ρ is the density of water, while g is the gravitational constant.

▶ plots and solution