Tricks with Power Series: extending the geometric series summation formula

Derive a power series representation of $\frac{x^2}{(7+x)^3}$ by starting with the geometric series representation of $\frac{1}{7+x}$, then differentiate twice to get an expression proportional to $\frac{1}{(7+x)^3}$, then multiply by x^2. (Stewart Calculus 8e. 11.9.13)

$$\frac{1}{7+x} = \frac{1}{7\left(1-\left(-\frac{x}{7}\right)\right)} = \frac{1/7}{1-\left(-\frac{x}{7}\right)} \to a = 1/7, \quad \to r = \frac{-x}{7}, \quad |r|<1 \to |x|<7, \quad -7<x<7:$$

$$= \sum_{n=0}^{\infty} \frac{1}{7}\left(\frac{-x}{7}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{7^{n+1}}$$

$$\frac{d}{dx}(7+x)^{-1} = -(7+x)^{-2}$$

$$\frac{d^2}{dx^2}(7+x)^{-1} = -(-2)(7+x)^{-3} = \frac{2}{(7+x)^3} \to \frac{1}{(7+x)^3} = \frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{7+x}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{7^{n+1}}\underbrace{\frac{d^2}{dx^2}x^n}$$

$$\frac{d}{dx}\left(nx^{n-1}\right) = n(n-1)x^{n-2}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n n(n-1)}{2\cdot 7^{n+1}} x^{n-2} = \sum_{n=2}^{\infty}\frac{(-1)^n n(n-1) x^{n-2}}{2\cdot 7^{n+1}}$$

first 2 terms are zero

$$\boxed{\frac{x^2}{(7+x)^3} = x^2\sum_{n=2}^{\infty}\frac{(-1)^n n(n-1) x^{n-2}}{2\cdot 7^{n+1}} = \sum_{n=2}^{\infty}(-1)^n\frac{n(n-1)}{2\cdot 7^{n+1}}x^n}$$

If instead of x^2 we had a different power, we would have to re-order the dummy index n. For example:

$$\frac{x}{(7+x)^3} = x\sum_{n=2}^{\infty}(-1)^n\frac{n(n-1)}{2\cdot 7^{n+1}}x^{n-2} = \sum_{n=2}^{\infty}(-1)^n\frac{n(n-1)}{2\cdot 7^{n+1}}x^{n-1}$$

let $m = n-1$
so $n = m+1$
$n+1 = m+2$
$n=2 \to m=1$

$$= \sum_{m=1}^{\infty}(-1)^{n-1}\frac{(m+1)m}{2\cdot 7^m}x^m$$

This same technique works for any functions of the form:

$$\frac{cx^p}{a+bx} \qquad p \geq 0, \text{ integer}$$