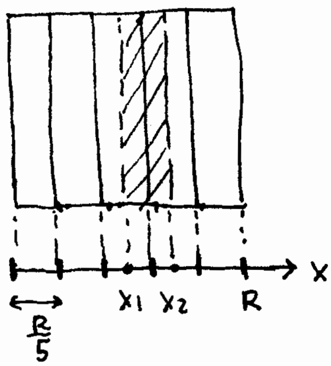
1-D probability distributions

Given that a dart does hit the dartboard from a random dart toss (all locations on dartboard equally likely to be hit), what is the probability that a dart hits:

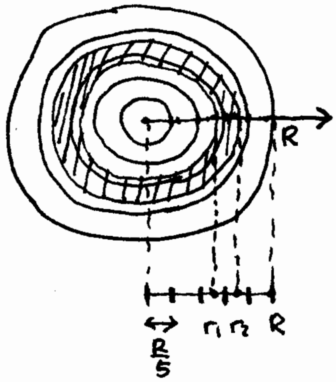
a strip between $x=x_1$ and $x=x_2$ on a square vertically striped dartboard (5 equal width strips)?

a ring between $r=r_1$ and $r=r_2$ on a circular dart board (5 equal radius increment rings)



horizontal position

probability = fractional area



radial position

Goal: $P(x_1 \le x \le x_2) = ?$ $\qquad\longleftrightarrow\qquad$ $P(r_1 \le r \le r_2) = ?$

Calculate:

$$\| \qquad \frac{\text{area (strip)}}{\text{area (square)}} = \frac{(x_2-x_1)R}{R^2} = \frac{x_2-x_1}{R}$$

$$= \frac{x}{R}\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} \underbrace{\frac{1}{R}}_{\rho(x)}\, dx$$

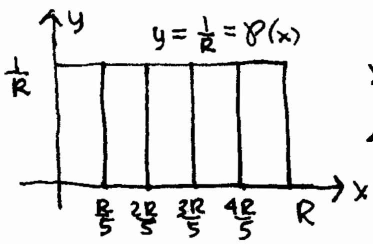
$$\| \qquad \frac{\text{area (ring)}}{\text{area (circle)}} = \frac{\pi r_2^2 - \pi r_1^2}{\pi R^2} = \frac{r_2^2 - r_1^2}{R^2}$$

$$= \frac{r^2}{R^2}\Big|_{r_1}^{r_2} = \int_{r_1}^{r_2} \underbrace{\frac{2r}{R^2}}_{\rho(r)}\, dr$$

$\rho(x)$ ⟵ probability distribution ⟶ $\rho(r)$

Check: (total prob. is 1)

$$\int_0^R \rho(x)\,dx = \int_0^R \frac{1}{R}\,dx = 1 \qquad \left(\begin{array}{c}\text{probability =}\\ \text{area under prob.}\\ \text{distribution curve}\end{array}\right) \qquad \int_0^R \rho(r)\,dr = \int_0^R \frac{2r}{R^2}\,dr = 1$$

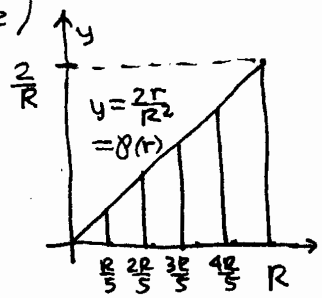


$x_i = \frac{i}{5}R,\ i=1..5$

$\Delta P_i = \left(\frac{R}{5}\right)\left(\frac{1}{R}\right) = \frac{1}{5}$

ΔP_i: $\frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}$, sum $= \frac{5}{5} = 1$

"uniform" prob. distribution



$r_i = \frac{i}{5}R,\ i=1..5$

$\Delta P_i = (r_i^2 - r_{i-1}^2)/R^2$

$= \left[\left(\frac{i}{5}R\right)^2 - \left(\frac{i-1}{5}R\right)^2\right]/R^2$

$= \frac{i^2-(i-1)^2}{25}$

$= \frac{2i-1}{25}$

ΔP_i: $\frac{1}{25}\ \frac{3}{25}\ \frac{5}{25}\ \frac{7}{25}\ \frac{9}{25}$, sum $= \frac{25}{25} = 1$

" biased towards larger r "

expected position ("average" position)? weight midpoints by strip probability

discrete:

$$\langle x\rangle = \frac{1}{5}\left(\frac{R}{10}\right) + \frac{1}{5}\left(\frac{3R}{10}\right) + \frac{1}{5}\left(\frac{5R}{10}\right) + \frac{1}{5}\left(\frac{7R}{10}\right) + \frac{1}{5}\left(\frac{9R}{10}\right)$$

$$= \frac{1}{5}\left(\frac{1+3+5+7+9}{10}\right)R = \frac{5}{10}R = \frac{R}{2} = .50R$$

$$= \sum_{i=1}^{5} x_i^* \Delta P_i \longrightarrow \lim_{n\to\infty} \sum_{i=1}^{n} x_i^* \rho(x_i^*)\Delta x$$

continuous limit:

$$= \int_0^R x\,\frac{1}{R}\,dx = \frac{x^2}{2R}\Big|_0^R = \frac{R}{2} \quad \text{(midpoint)}$$

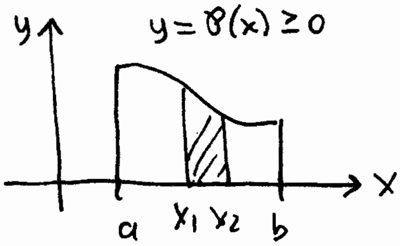
$$\langle r\rangle = \frac{1}{25}\left(\frac{R}{10}\right) + \frac{3}{25}\left(\frac{3R}{10}\right) + \frac{5}{25}\left(\frac{5R}{10}\right) + \frac{7}{25}\left(\frac{7R}{10}\right) + \frac{9}{25}\left(\frac{9R}{10}\right)$$

$$= 33/50R = .66R$$

$$= \sum_{i=1}^{5} r_i^* \Delta P_i \to \lim_{n\to\infty} \sum_{i=1}^{n} r_i^* \rho(r_i^*)\Delta r$$

$$= \int_0^R r\left(\frac{2r}{R^2}\right)dr = \frac{2}{3}\frac{r^3}{R^2}\Big|_0^R = \frac{2}{3}R$$

1-D probability distributions (2)

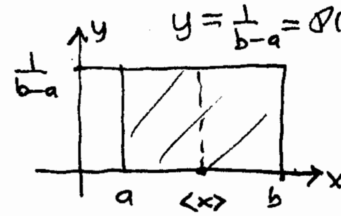


$$P(x_1\leq x\leq x_2)=\int_{x_1}^{x_2}\rho(x)\,dx$$

$$P(a\leq x\leq b)=\int_a^b \rho(x)\,dx=1$$

$[a,b]\to$ semi-infinite or infinite intervals also

a) uniform distribution:



$$\langle x\rangle=\int_a^b x\left(\frac{1}{b-a}\right)dx=\frac{x^2}{2(b-a)}\Big|_a^b$$

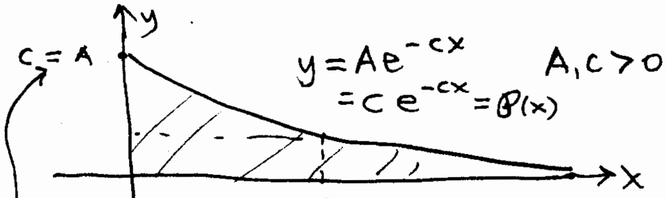
$$=\frac{b^2-a^2}{2(b-a)}=\frac{(b-a)(b+a)}{2(b-a)}=\frac{a+b}{2}$$

(midpoint)

b) semi-infinite "Poisson" distribution

$$\int_0^\infty Ae^{-cx}\,dx=-\frac{A}{c}e^{-cx}\Big|_0^\infty$$

$$=\frac{A}{c}-\underbrace{\text{"}\frac{A}{c}e^{-\infty}\text{"}}_{0}=\frac{A}{c}=1\to A=c$$



$\frac{1}{c}=\mu \to \rho(x)=\frac{1}{\mu}e^{-\frac{x}{\mu}}$

$\parallel$

$\langle x\rangle$

$$\langle x\rangle=\int_0^\infty x\,ce^{-cx}\,dx=-\left(\frac{cx+1}{c}\right)e^{-cx}\Big|_0^\infty=\frac{1}{c}\Big(1-\underbrace{\lim_{x\to\infty}\frac{x}{e^{x}}}_{0\text{ by L'Hopital's rule}}\Big)=\frac{1}{c}\equiv\mu$$

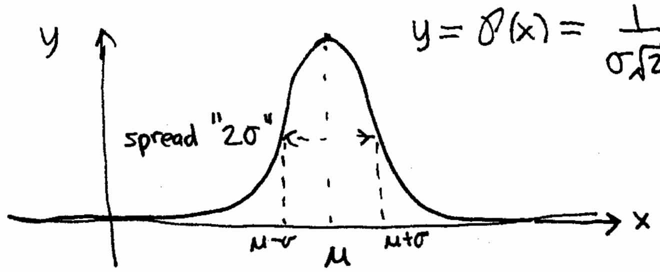
c) infinite normal distribution

$$y=\rho(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$-\infty\leq x\leq\infty$

$\hookrightarrow$ sometimes $0\leq x\leq\infty$

(since contribution for $x\leq 0$ negligible.)



tail sometimes ignored if $x\geq 0$ variable

$$\int_{-\infty}^{\infty}\rho(x)=1$$

(we cant show this but Calc3 can)

probability of being one "standard deviation" from the "average" value μ:

$$\int_{\mu-\sigma}^{\mu+\sigma}\rho(x)\,dx \underset{}{\overset{\text{maple}}{=}} \dots \overset{\text{subs}(\sigma=1,\mu=0)}{=} .6829$$

$\parallel$

$u=\frac{x-\mu}{\sigma}$

variable change

$$\int_{-1}^{1}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\,du$$

"standard normal curve" in units of standard deviation from expected value