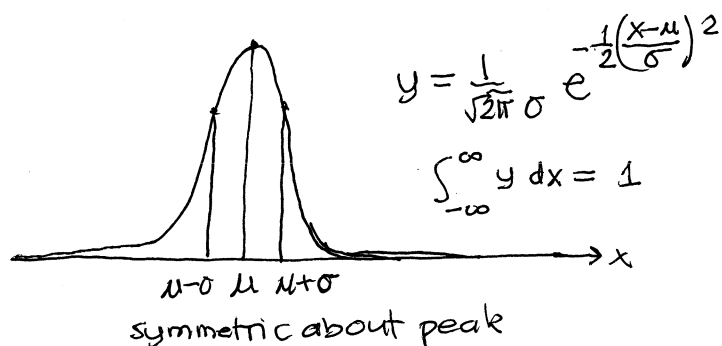


the bell curve (normal distribution): an important example of definite integral change of variable



The bell curve describes the distribution of values of many physical variables, like the height of adult males in a given population — there is a peak around some average value, some spread about that peak and then it falls off rapidly away from that peak.

If we move the peak to the origin and measure the variable x in units of the so called standard deviation σ , we get a standard bell curve. Areas under a general bell curve correspond directly to areas under the standard bell curve, and so probability questions can instead be answered by referring them to that standard curve: $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}$, $u = \frac{x-\mu}{\sigma}$ $du = \frac{dx}{\sigma}$

The probability that x takes a value in the interval $a \leq x \leq b$ is

$$P(a \leq x \leq b) = \int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{dx}{\sigma} = \int_{x=0}^{x=b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$= \int_{u_a}^{u_b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du, \text{ where } u_a = \frac{a-\mu}{\sigma}, u_b = \frac{b-\mu}{\sigma}$$

In the old days the area function for this function graph was tabulated so that probability questions could be answered — since technology did not exist to calculate these areas.

Now we have the special function " $\frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$ " ("error function") defined to be the area function zeroed at the origin, so that

$$P(u_a \leq u \leq u_b) = \int_{u_a}^{u_b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = \frac{1}{2} \operatorname{erf}\left(\frac{u_b}{\sqrt{2}}\right) - \frac{1}{2} \operatorname{erf}\left(\frac{u_a}{\sqrt{2}}\right)$$

The factors of 2 require an explanation for another day.

[Another change of variable: $\frac{1}{2} \int e^{-\frac{1}{2}u^2} du = \int e^{-v^2} dv$ if $v = u/\sqrt{2}$ google it]