changing variables in an integral (1)

$$\int_0^a x^3\sqrt{a^2-x^2}\,dx = \int_{x=0}^{x=a\,\to\,u=0} \underbrace{x^2}_{a^2-u}\ \underbrace{(a^2-x^2)^{1/2}}_{u}\ \underbrace{(x\,dx)}_{-\frac{du}{2}} = -\frac{1}{2}\int_{a^2}^{0}(a^2-u)u^{1/2}\,du$$

switch

$x=0 \to u=a^2$

$u = a^2-x^2 \to \frac{du}{dx} = -2x$

$x^2 = a^2-u$

$du = -2x\,dx$

$-\frac{du}{2} = x\,dx$

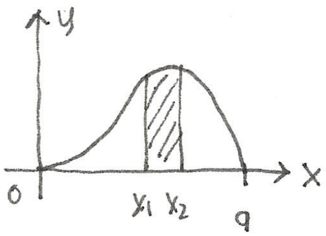
$$= \frac{1}{2}\int_0^{a^2}(a^2-u)u^{1/2}\,du$$

$$= \frac{1}{2}\int_0^{a^2}(a^2u^{1/2}-u^{3/2})\,du$$

$$= \frac{1}{2}\left(a^2\left(\frac{2u^{3/2}}{3}\right) - \frac{2}{5}u^{5/2}\right)\Big|_0^{a^2}$$

$$= \frac{a^2a^3}{3} - \frac{a^5}{5} = \frac{2}{15}a^5$$

you can replace expressions only by new expressions which equal the old ones: x and dx must be re-expressed in terms of u and du using equations between them.



Dividing this nonnegative function (between 0 and a) by its area leads to a new function with unit area, hence a probability distribution function

$$\rho(x) = \frac{15}{2a^5}x^3\sqrt{a^2-x^2}, \quad P(x_1\le x\le x_2) = \int_{x_1}^{x_2}\rho(x)\,dx$$

With $a = 12$ gal, this could model the probability that a self-serve gas customer adds x gals to the car gas tank: it is not very probable that a small amount will be added but much more likely that a large fraction of the tank will be added. We could use this model for all cars with 12 gal tanks.

On the other hand the fractional tank capacity variable $z = \frac{x}{a}$ is perhaps more useful: what fraction z of the full tank is added? We can transform the integral & probability distribution to correspond to this new variable by changing the variable:

$$P(x_1\le x\le x_2) = \int_{x=x_1}^{x=x_2}\frac{15}{2a^5}\underbrace{x^3}_{(az)^3 = a^3z^3}\ \underbrace{(a^2-x^2)^{1/2}}_{\ldots}\ \underbrace{dx}_{a\,dz} = \int_{z=z_1}^{z=z_2}\frac{15}{2a^5}(a^3z^3)\,a\,(1-z^2)^{1/2}\,a\,dz$$

$z = \frac{x}{a} \to z_i = \frac{x_i}{a}$

$x = az \to dx = a\,dz$

$a^2-(az)^2$

$a^2-a^2z^2$

$a^2(1-z^2)$

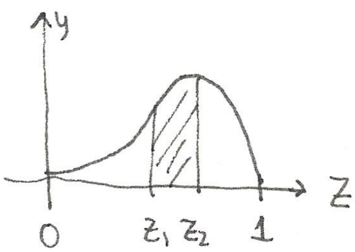
$[a^2(1-z^2)]^{1/2}$

$\underbrace{[a^2]^{1/2}}_{a}(1-z^2)^{1/2}$

rules of exponents!!

$$= \int_{z_1}^{z_2}\frac{15\cancel{a^5}}{2\cancel{a^5}}z^3(1-z^2)^{1/2}dz = \int_{z_1}^{z_2}\underbrace{\frac{15}{2}z^3(1-z^2)^{1/2}}_{\text{new probability density function}}dz$$

$$= \underbrace{P(z_1\le z\le z_2)}_{\text{probability that fraction of tank added is between } z_1 \text{ and } z_2}$$



In fact with fractional capacity we can apply to model to all size gas tanks so it is more useful too.

we cannot do, Maple can

Expected value: $\mu = \frac{15}{2a^5}\int_0^a x\,x^3\sqrt{a^2-x^2}\,dx = \frac{15}{2a^5}\int_0^a x^4\sqrt{a^2-x^2}\,dx = \cdots = \left(\frac{15\pi}{64}\right)a \approx 0.736\,a$

The average amount of gas added is nearly 3/4 a tank.

Probability of adding less than half a tank: $P(0\le x\le a/2) = \frac{15}{2a^5}\int_0^{a/2}x^3\sqrt{a^2-x^2}\,dx = \ldots \approx 0.107$ (about 11%)