binomial expansion ("expand" in a Taylor series)

$$(a+b)^n = a^n + na^{n-1}b + \dots + \underbrace{\frac{n!}{m!(n-m)!}}_{\binom{n}{m}} a^{n-m}b^m + \dots nab^{n-1} + b^n, \quad n \text{ integer}$$

$\binom{n}{m}$ binomial coefficient.

$\downarrow$ integer

$$f(x) = (1+x)^n = 1 + nx + \dots + nx^{n-1} + x^n \quad \text{terminating Taylor series}$$

$$= T_n(x)$$

(all polynomials equal their Taylor series expansions about any center a.)

$\downarrow$ any real #

$f(x) = (1+x)^k$ $\qquad f(0) = 1$

$f'(x) = k(1+x)^{k-1}$ $\qquad f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}$ $\qquad f''(0) = k(k-1)$

$f^{(3)}(x) = \underbrace{k(k-1)(k-2)}_{3 \text{ factors}}(1+x)^{k-3}$ $\qquad f^{(3)}(0) = k(k-1)(k-2)$

$f^{(n)}(x) = \underbrace{k(k-1)(k-2)\cdots(k-(n-1))}_{n \text{ factors}}(1+x)^{k-n}$ $\qquad f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1)$

$$\boxed{(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n}$$

n factors

"generalized" binomial coefficient $\binom{n}{k}$

terminates for positive integers (polynomials)

$$(1+x)^4 = 1 + \frac{4}{1}x + \frac{4\cdot 3}{1\cdot 2}x^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}x^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}x^4$$

$$= 1 + 4x + 6x^2 + 4x^3 + x^4$$

$$(1+x)^{-4} = 1 + \frac{(-4)}{1}x + \frac{(-4)(-5)}{1\cdot 2}x^2 + \frac{(-4)(-5)(-6)}{1\cdot 2\cdot 3}x^3 + \dots$$

infinite series for all other cases

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|k(k-1)(k-2)\dots(k-(n-1))(k-n)|}{(n+1)!} \cdot \frac{n!}{|k(k-1)(k-2)\dots(k-(n-1))|} \frac{|x|^{n+1}}{|x|^n}$$

$$= \frac{|k-n|}{n+1}|x| \xrightarrow{n\to\infty} \boxed{|x| < 1} \text{ for convergence}$$

$|x| = 1$? depends on k (not so important anyway)

UTILITY: $|x| \ll 1$ converges much quicker

In fact linear approx often useful:

$$\left.\begin{aligned} (1+x)^k &\approx 1 + kx \\ \frac{1}{(1+x)^k} = (1+x)^{-k} &\approx 1 - kx \end{aligned}\right\} \text{ for } |x| \ll 1 \text{ (small!)}$$

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example extending to $\dfrac{ax^q}{(b+cx^p)^k}$: $\quad f(x) = \dfrac{1}{\sqrt{4-x^2}} = \dfrac{1}{\sqrt{4(1-\frac{x^2}{4})}} = \dfrac{1}{2}\left(1 + \underbrace{\left(\frac{-x^2}{4}\right)}_{\text{replaces } x}\right)^{\overbrace{-1/2}^{k}}$