

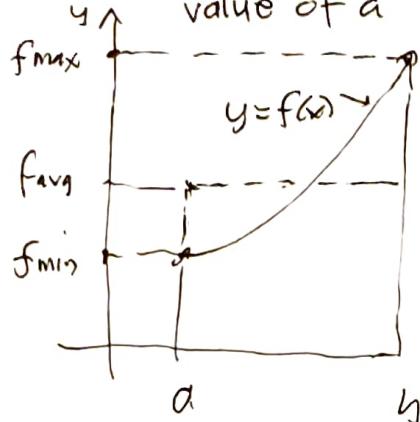
8.5b

probability

①

CAUTION

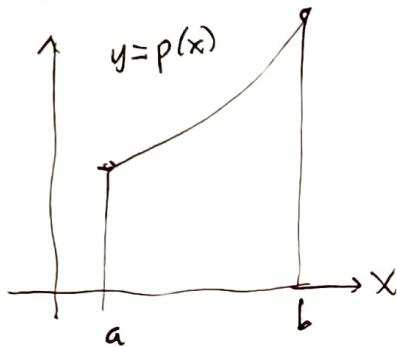
Don't confuse this averaging process with the average value of a function.



$$f_{\text{avg}} = \frac{\int_a^b f(x) dx}{b-a} = \frac{\int_a^b y(x) dx}{b-a}$$

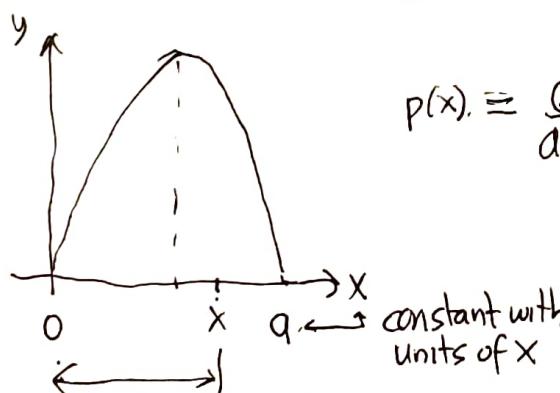
here we are averaging the [y-values] over an interval on the y-axis, resulting in an average value "somewhere in the middle"

In probability we are averaging the [x-values] over an interval on the x-axis with the average value "somewhere in the middle".

Dimensions

probability is a pure number = "dimensionless"
only x carries physical dimensions, units of whatever physical variable it represents,
like length, time, weight
although sometimes it too is dimensionless.

Often we can transform x to a new dimensionless variable with a "standard distribution"

Example

$$f(x) = x(a-x) \rightarrow A \int_0^a ax - x^2 dx = \left. \frac{ax^2}{2} - \frac{x^3}{3} \right|_0^a = a^3/6$$

$$p(x) = \frac{6}{a^3} x(a-x), \quad 0 \leq x \leq a$$

let $u = \frac{x}{a}$ = fractional distance from 0 to x compared to a
units cancel

transform probability integral:

$$\rightarrow du = \frac{dx}{a} \text{ or } dx = a du$$

$$\begin{cases} x=0 \rightarrow u=\frac{0}{a}=0 \\ x=a \rightarrow u=\frac{a}{a}=1 \end{cases} \quad \underbrace{0 \leq u \leq 1}_{\text{new "standard variable interval"}}$$

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Probability

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$$P(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} \frac{6}{a^3} x(a-x) dx$$

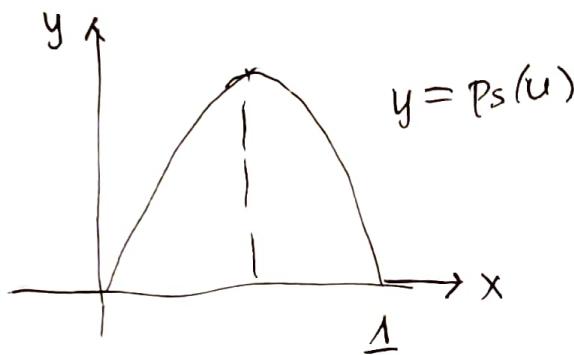
$u_1 = \frac{x_1}{a}$ $u_2 = \frac{x_2}{a}$

$$= \int_{u_1}^{u_2} \frac{6}{a^3} (ax)(a)(1-u) a du = \int_{u_1}^{u_2} \underbrace{6u(1-u)}_{p_s(u)} du$$

new standard distribution

$$= P(u_1 \leq u \leq u_2)$$

Equivalent to setting $a=1$ (and measuring x in multiples of a)



So to evaluate probability quantities, we first restate questions in terms of the new variable & evaluate using the new "standard distribution".

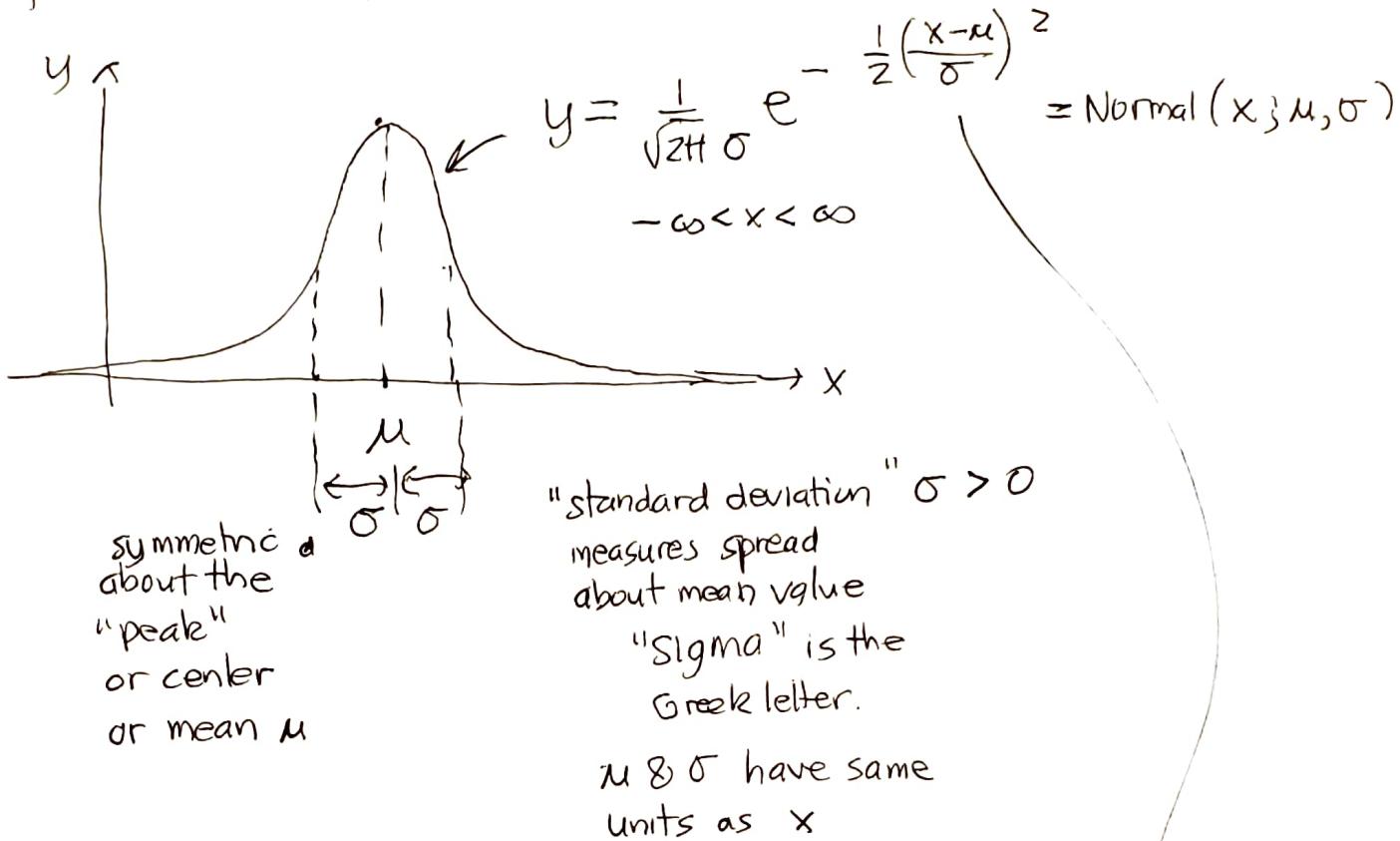
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Probability

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The normal distribution = "bell curve"

This is an infinite interval probability distribution usually applied to a finite closed interval by ignoring the rapidly decreasing "tails" of the distribution.



μ only shifts the graph horizontally without changing the shape and can vanish while σ provides a natural unit to measure x values relative to the mean

$$P(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$\uparrow \quad \downarrow$
 $e^{-\frac{1}{2}u^2}$ du cancel

$$= \int_{u_1}^{u_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$\text{Normal}(u; 0, 1)$ standard distribution

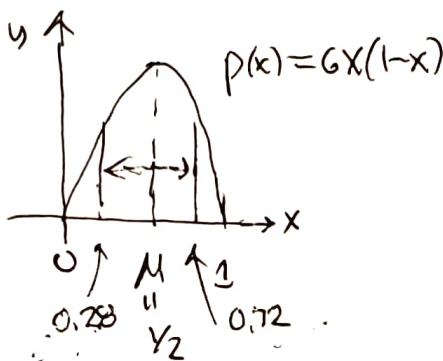
u measures the number of standard deviations from the mean

Aside The book doesn't mention σ but it can be defined for any distribution

$\sigma^2 = \langle (x-\mu)^2 \rangle = \text{average squared distance of } x \text{ from the mean } \mu$

↑ ↑
guarantees that
 σ has same units as x and μ .

Example.



$$\sigma^2 = \int_0^1 (x-\frac{1}{2})^2 6x(1-x) dx$$

$$= \frac{1}{20} \rightarrow \sigma = \sqrt{\frac{1}{20}} \approx 0.224$$

Maple

(easy to do)
by hand

$$\begin{aligned} & 0.500 & & \cdot 0.00 \\ & -0.224 & + & \cdot 0.224 \\ & \hline & .276 & .724 \\ & \mu - \sigma & & \mu + \sigma \end{aligned}$$

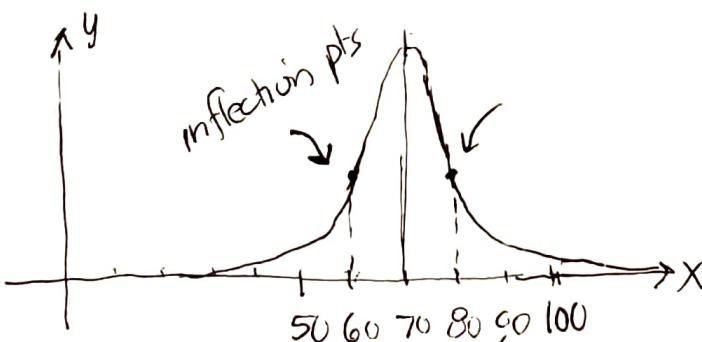
roughly $1/4$ of interval -
on either side.

But this is just for your optional background

Normal distribution example.

A distribution of test grades with mean $\mu = 70$ and standard deviation $\sigma = 10$ so:

$$\begin{aligned} p(x) &= \text{Normal}(x; 70, 10) \\ &= \frac{1}{\sqrt{2\pi}(10)} e^{-\frac{1}{20}(x-70)^2} \end{aligned}$$



(see Maple worksheet)

about $2/3$ of all the grades are within 10 pts of the average.

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Probability

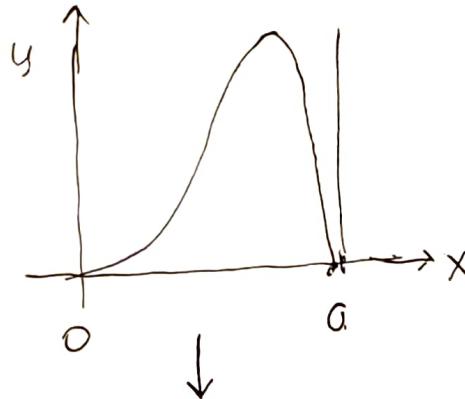
(5)

Example: Gas Tank filling

When a person "fills up" a gas tank of capacity a gallons, how much is added to the tank?

Let x = amount of gas in gals during a fillup event

$$0 \leq x \leq a$$



start with $f(x) = x^3 \sqrt{a^2 - x^2}$

calculate area

$$A = \int_0^a x^3 \sqrt{a^2 - x^2} dx = \frac{2a^5}{15}$$

$$p(x) = \frac{15}{2a^5} x^3 \sqrt{a^2 - x^2}$$

$$u = \frac{x}{a} \text{ dimensionless}$$

= fraction of tank filled

$$du = \frac{dx}{a} \rightarrow dx = a du$$

$$\int_{x_1}^{x_2} \frac{15}{2a^5} x^3 \sqrt{a^2 - x^2} dx$$

$\underbrace{\frac{1}{a} \sqrt{1 - (\frac{x}{a})^2} a du}$
 $\underbrace{(au)^3}_{(au)^3}$

$$= \int_{u_1}^{u_2} \frac{15}{2a^5} u^3 \sqrt{1 - u^2} a du$$

$$= \int_{u_1}^{u_2} \frac{15}{2} u^3 \sqrt{1 - u^2} du$$

standard distribution

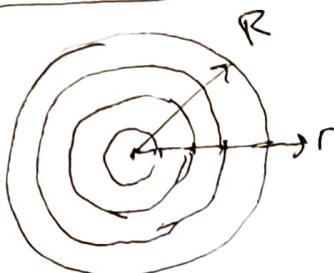
equivalent to setting $a = 1$.

8.5b

Probability

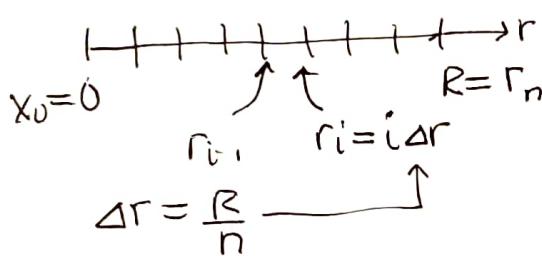
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Fun example: continuous \rightarrow discrete \rightarrow continuous (full circle)

Target rings

$n=4$ rings
equally spaced
 $0 \leq r \leq R$

Riemann approach:



fractional areas:
= corresponding probabilities

$$\frac{\Delta A_i}{A} = \frac{\pi(2i-1)(R/n)^2}{\pi R^2} = \frac{(2i-1)}{n^2} = p_i$$

$$n=4 : \frac{x_i}{R} : \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}$$

$$p_i : \frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16} \xrightarrow{\text{sum}} \frac{1+3+5+7}{16} = \frac{16}{16} = 1 \quad \checkmark$$

\uparrow probabilities of hitting i th ring

What is the expected value of the ring number $i = 1, 2, 3, 4$?

$$\begin{aligned} \langle \text{ring\#} \rangle &= \sum_{i=1}^4 i \cdot p_i = 1\left(\frac{1}{16}\right) + 2\left(\frac{3}{16}\right) + 3\left(\frac{5}{16}\right) + 4\left(\frac{7}{16}\right) \\ &= \frac{1+6+15+28}{16} = \frac{50}{16} = \frac{25}{8} = 3.125 \end{aligned}$$

Interpretation: Throw 1000 darts randomly at the target and hit it. Average all the ring numbers where they land, the result is 3.125.

Consider a circular target with n equally distributed concentric rings.

If a dart is "randomly" thrown at the target and hits it, what is the probability that it hits one of the rings?

It seems clear that the probability should equal the fractional area of each ring compared to the total area.

Area of i th ring:

$$\begin{aligned} \Delta A_i &= \pi(r_i^2 - r_{i-1}^2) \\ &= \pi(i^2 \Delta r^2 - (i-1)^2 \Delta r^2) \\ &= \pi(i^2 - (i-1)^2) \Delta r^2 \\ &\quad \cancel{i^2} \cancel{(i^2 - 2i + 1)} \\ &= \pi(2i-1) \Delta r^2 \end{aligned}$$

$$\text{total area } A = \sum_{i=1}^n \Delta A_i = \pi R^2$$

8.5b)

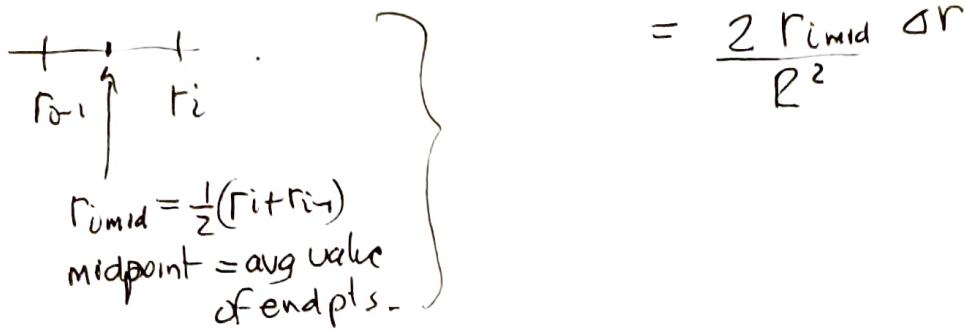
Probability

(7)

Suppose we keep increasing the number of equally spaced rings (Riemann!) so n is very large.

$$p_i = \frac{\Delta A_i}{A} = \frac{\pi(r_i^2 - r_{i-1}^2)}{\pi R^2} = \frac{(r_i + r_{i-1})(r_i - r_{i-1})}{R^2}$$

$$= 2r_{i\text{mid}} \Delta r$$



$$\sum_{i=1}^n p_i = \sum_{i=1}^n \frac{2r_{i\text{mid}} \Delta r}{R^2} \quad \text{or}$$

$$\underbrace{\sum_{i=1}^n p_i}_{\text{total probability}} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2r_{i\text{mid}} \Delta r}{R^2} \quad \text{or} = \int_0^R \frac{2r}{R^2} dr = p(r)$$

$r_i^* = \text{midpoint}$

continuous distribution

$$= \left. \frac{r^2}{R^2} \right|_0^R = \frac{R^2}{R^2} = 1 \quad \checkmark$$

$$\text{In fact } \int_{r_1}^{r_2} \frac{2r}{R^2} dr = \left. \frac{r^2}{R^2} \right|_{r_1}^{r_2} = \frac{r_2^2 - r_1^2}{R^2} = \frac{\pi(r_2^2 - r_1^2)}{\pi R^2} = \frac{\text{fractional area}}{\text{area}}$$

If we are not interested in rings, but simply radial distance from the center, we can just use this continuous distribution we started with before discretizing it by imposing rings.

$$\langle r \rangle = \int_0^R r \left(\frac{2r}{R^2} \right) dr = \int_0^R \frac{2r^2}{R^2} dr = \left. \frac{2}{3} \frac{r^3}{R^2} \right|_0^R = \frac{2}{3} \frac{R^3}{R^2} = \frac{2}{3} R \approx 0.67 R$$

The expected value is $2/3$ the target radius.

If we throw 1000 darts and average their radius, we will find about $0.67 R$.