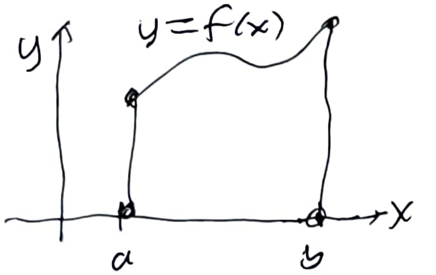
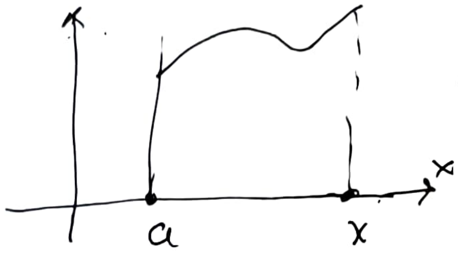
8.1b arclength functions





variable right endpoint

fixed
left endpoint

$$L = \int_a^b \sqrt{1+f'(x)^2}\,dx$$

$$S(x) = \int_a^x \sqrt{1+f'(t)^2}\,dt$$

dummy variable replacing x.

$$\frac{ds(x)}{dx} = \underbrace{\sqrt{1+f'(x)^2}}_{}$$ by fund thm of calculus.

any antiderivative measures changes in arclength

the arbitrary choice of ~~refererence~~ reference point fixes the zero of this signed curvature function.

$S(a) = 0$ arclength is measured relative to point $x=a$ on graph

$S(x) > 0$ for $x > a$
$S(x) < 0$ for $x < a$ } signed arclength

Independent of this choice, differences of the arclength function give the arclength between any two points on the graph

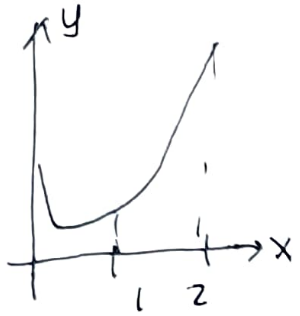
$$\int_{x_1}^{x_2} \sqrt{1+f'(x)^2}\,dx = S(x_2) - S(x_1)$$

Who cares? we need this tool to study motion in the plane where the arclength function measures distance traveled along a path, as long as the path is the graph of a function.

For arbitrary curves, we wait till Chapter 10.

$\boxed{8.1b}$ arclength functions

example perfect square case from last time



$$y = x^2 - \frac{1}{8}\ln x$$

$$\frac{dy}{dx} = 2x - \frac{1}{8x}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(2x - \frac{1}{8x}\right)^2$$

$$= 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x} = \left(2x + \frac{1}{8x}\right)^2$$

$$+\frac{1}{2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\left(2x + \frac{1}{8x}\right)^2} = 2x + \frac{1}{8x}$$

$$s(x) = \int_1^x 2t + \frac{1}{8t}\,dt = t^2 + \frac{1}{8}\ln t\,\Big|_1^x$$

$$= x^2 + \frac{1}{8}\ln x - 1^2 - \underbrace{\frac{1}{8}\ln 1}_{0} = \underbrace{x^2 + \frac{1}{8}\ln x}_{\text{antiderivative}} - 1$$

forces $s(1) = 0$

$$s(2) = 2^2 + \frac{1}{8}\ln 2 - 1 = 3 + \frac{1}{8}\ln 2 > 0$$
$$\approx 16.05$$

$$s\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^2 + \frac{1}{8}\underbrace{\ln\frac{1}{2}}_{-\ln 2} - 1 = \frac{1}{4} - 1 - \frac{1}{8}\ln 2 = -\frac{3}{4} - \frac{1}{8}\ln 2 < 0$$
$$\approx -5.48$$

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only preferred pt on graph: minimum

$$0 = \frac{dy}{dx} = 2x - \frac{1}{8x} = 0 \longrightarrow x^2 = \frac{1}{16} \rightarrow x = \frac{1}{4}$$

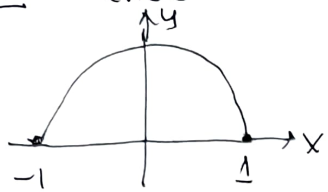
$$s\left(\tfrac{1}{4}\right) = \left(\tfrac{1}{4}\right)^2 + \frac{1}{8}\ln\left(\tfrac{1}{4}\right)^{-1} = \frac{1}{16} - 1 - \frac{1}{8}\ln 2^2 = -\frac{15}{16} - \frac{1}{4}\ln 2$$

$$\underbrace{s(x) - s\left(\tfrac{1}{4}\right)}_{} = x^2 + \frac{1}{8}\ln x - 1 + \frac{15}{16} + \frac{1}{4}\ln 2 = x^2 + \frac{1}{8}\ln x - \frac{1}{16} + \frac{1}{4}\ln 2$$

arclength function zeroed at $x = 1/4$.

8.1b arclength functions

example circle



$y = \sqrt{1-x^2}$

$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1-x^2}}$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1-\cancel{x^2}+\cancel{x^2}}{1-x^2} = \frac{1}{1-x^2}$

$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{\sqrt{1-x^2}}$

$x=-1$ reference pt?

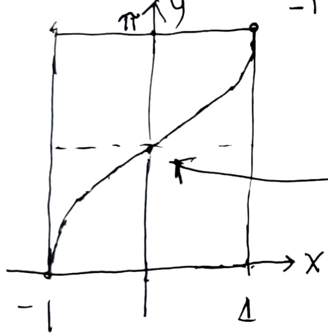
$S(x) = \int_{-1}^{x} \frac{1}{\sqrt{1-t^2}}\,dt$

compare with area accumulation $A(x) = \int_{-1}^{x} \sqrt{1-x^2}\,dx$

both ugly because x & y are "rectangular" coords incompatible with circles! $\boxed{\text{Chapter 10 polar coords!}}$

Maple

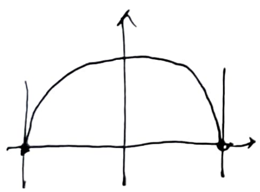
$S(x) = \arcsin t \Big|_{-1}^{x} = \arcsin x - \underbrace{\arcsin(-1)}_{-\frac{\pi}{2}} = \arcsin x + \frac{\pi}{2}$



$\frac{ds}{dt}(x) = \frac{1}{\sqrt{1-x^2}} \longrightarrow +\infty$ as $x \to -1^+$ or $x \to 1^-$

slope 1

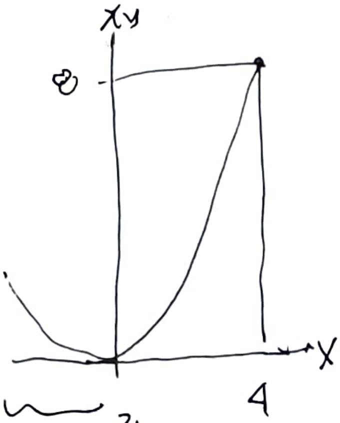
$s(1) = \arcsin 1 + \frac{\pi}{2}$
$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$



vertical tan lines at endpoints means arclength increases very quickly as function of x there

8.1b arclength functions

example



$y = |x|^{3/2}$

if we extend to an even function

$$y = x^{3/2}, \quad \text{from } x=0, \quad x \geq 0.$$

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{9}{4}x}$$

if $x \geq 0$!

$|x|$ otherwise

$$S(x) = \int_0^x \sqrt{\underbrace{1+\tfrac{9}{4}t}}\, dt \qquad \tfrac{4}{9}du$$

$$u = 1+\frac{9}{4}t$$

$$du = \frac{9}{4}dt$$

$$\int u^{1/2}\, \frac{4}{9}du = \frac{4}{9}\frac{u^{3/2}}{3/2} + C = \frac{8}{27}\left(1+\frac{9}{4}t\right)^{3/2} + C$$

$$= \frac{8}{27}\left[\left(1+\frac{9}{4}x\right)^{3/2} - 1\right] \qquad x \geq 0!$$

$S(-4) = ?$ careful

$$\sqrt{1+\frac{9}{4}|x|} = \sqrt{\underbrace{1-\tfrac{9}{4}x}}, \quad x \leq 0$$

$$u = 1-\frac{9}{4}x$$

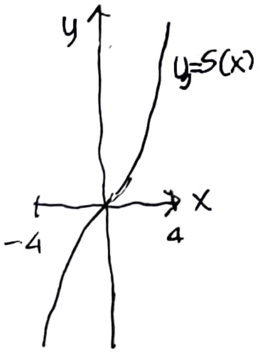
$$du = -\frac{9}{4}dx$$

$$\int u^{1/2}\left(-\frac{4}{9}du\right) = -\frac{4}{9}\frac{u^{3/2}}{3/2} = -\frac{8}{27}\left(1-\frac{9}{4}x\right)^{3/2}$$

$$S(x) = \int_0^x \sqrt{1-\frac{9}{4}t}\, dt = -\frac{8}{27}\left(1-\frac{9}{4}t\right)^{3/2}\Bigg|_0^x + \frac{8}{27}$$

$$= \frac{8}{27}\left(1-\left(1-\frac{9}{4}x\right)^{3/2}\right) \qquad x \leq 0$$

We are forced to a piecewise function.



This is an odd function, since $|x|^{3/2}$ is even.

$$S(-4) = -S(4)$$

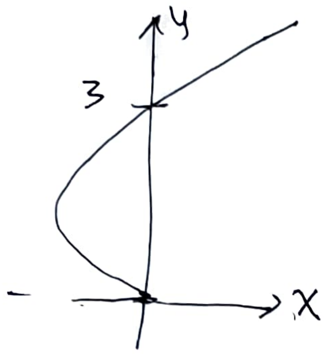
$$S(x) = \operatorname{sgn}(x) \cdot \frac{8}{27}\left[\left(1+\frac{9}{4}|x|\right)^{3/2} - 1\right]$$

arclength functions

class exercise : 8.1.13

$x = \frac{1}{3} y^{1/2}(y-3)$ perfect square trick

reference point $y=0$, $y \geq 0$. find $s(y)$.



compare $s(y)$ with $f(y) = \frac{1}{3} y^{1/2}(y-3)$. Interesting, no?