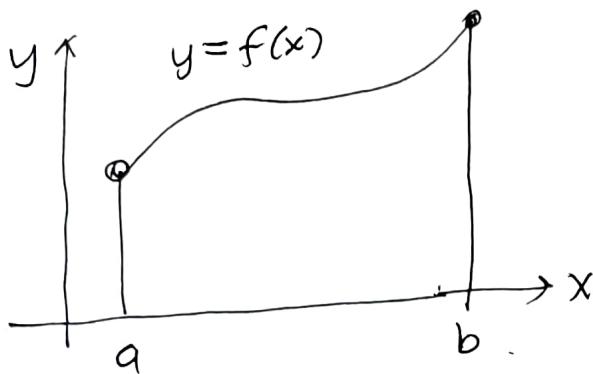


8.1a

arc length of curve segments

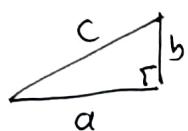
①

How do we define the length of a plane curve?

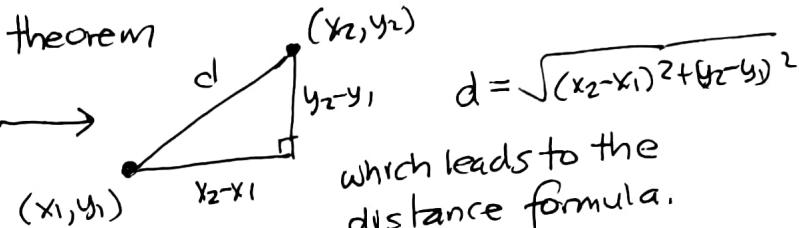


We apply the Riemann approach, zooming in to approximate the length of a small increment of the curve and setting up an integral formula for the limiting result.

We start from the Pythagorean theorem

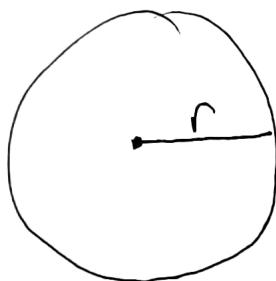


$$c = \sqrt{a^2 + b^2}$$



which leads to the distance formula.

The only other curve for which we have a length formula



$$C = 2\pi r$$

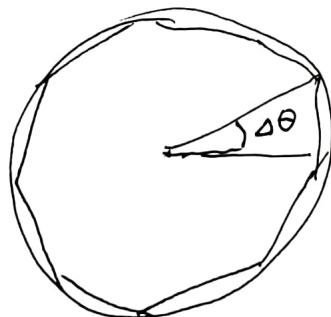
or an arc



$$L = \frac{\Delta\theta}{2\pi} (2\pi r) \\ = r\Delta\theta$$

But these require calculus to derive.

Archimedes:



limit of perimeter of regular n-sided polygon

leads to approximation for π

$$\Delta\theta = \frac{2\pi}{n}$$

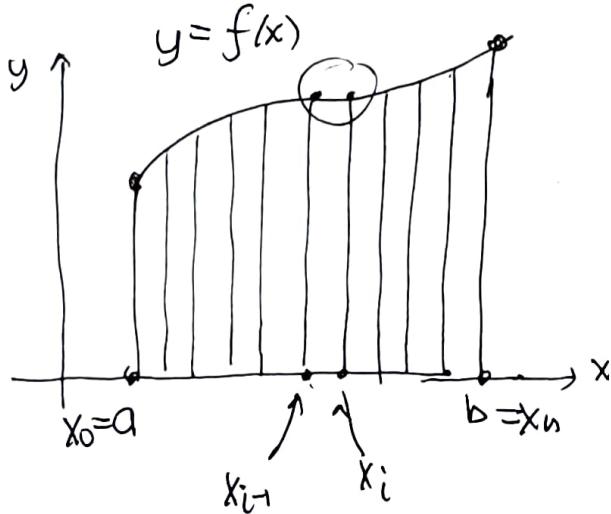
$$2 \sin \frac{\Delta\theta}{2} \xrightarrow{n \rightarrow \infty} n \cdot \left(2 \sin \left(\frac{2\pi}{2n}\right)\right) \xrightarrow{n \rightarrow \infty} 2\pi$$

But we can't use trig because it is based on knowing the circumference of the unit circle!

8.1a)

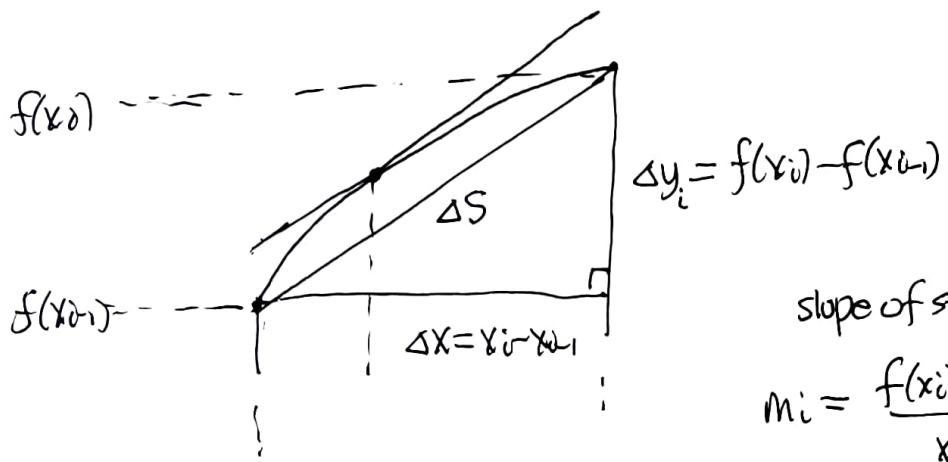
arc length of a curve segment

(2)



$$\Delta x = \frac{b-a}{n}$$

Zoom in



slope of secant line:

$$m_i = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x}$$

$$x_{i-1} \quad x_i^* \quad x_i$$

||

$f'(x_i^*)$ for some $x_{i-1} \leq x_i^* \leq x_i$
mean value theorem

$$\text{so } \Delta y_i = f'(x_i^*) \Delta x$$

$$\begin{aligned} (\Delta s_i)^2 &= (\Delta x)^2 + (\Delta y_i)^2 = (\Delta x)^2 + (f'(x_i^*))^2 (\Delta x)^2 \\ &= (\Delta x)^2 (1 + (f'(x_i^*))^2) \end{aligned}$$

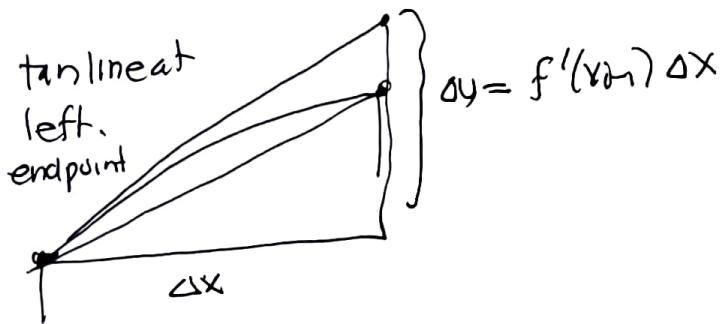
$$\begin{aligned} \Delta s_i &= \Delta x \sqrt{1 + (f'(x_i^*))^2} \\ L &\approx \sum_{i=1}^n \Delta s_i \rightarrow L = \lim_{n \rightarrow \infty} \sum_{i=1}^n \underbrace{\sqrt{1 + (f'(x_i^*))^2}}_{\rightarrow \text{integrand}} \Delta x \\ \Delta x &= \boxed{\begin{aligned} &\int_a^b \sqrt{1 + f'(x)^2} dx \\ &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \end{aligned}} \end{aligned}$$

8.1a

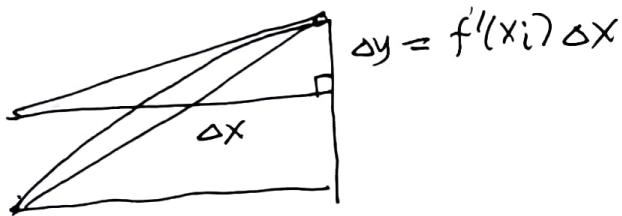
arc length of curve segment

(3)

we don't even need the mean value theorem



tan line at right endpoint



but as $\Delta x \rightarrow 0$, as long as the curve is differentiable, both tangent lines squeeze to the secant line so either is a good approximation to the secant line length which in turn approximates ds .

Remark

$$L = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{x=a}^{x=b} \sqrt{1 + \frac{dy^2}{dx^2}} dx$$

$$\sqrt{(1 + \frac{dy^2}{dx^2}) dx^2}$$

$$= \sqrt{dx^2 + dy^2} = ds$$

this is just the limiting version of Pythagorean thm!

Warning

$$\int \underbrace{\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}$$

putting a sqrt around any expression means it is rare to be able to find antiderivatives

ALL exactly integrable exercises rely on very special functions.

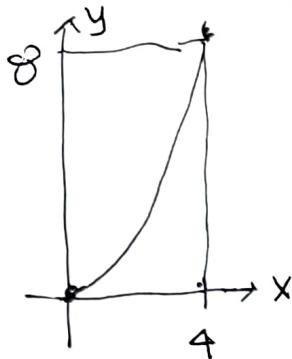
8.19

arclength of curve segment

(4)

example

$$y = x^{3/2}, \quad 0 \leq x \leq 4$$



$$\frac{dy}{dx} = \frac{3}{2}x^{1/2} \rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{9}{4}x$$

$$L = \int_0^4 \sqrt{1 + \frac{9}{4}x} dx \quad \text{easy u-sub}$$

$$u = 1 + \frac{9}{4}x$$

$$du = \frac{9}{4}dx$$

$$dx = \frac{4}{9}du$$

$$\begin{cases} x = u \rightarrow u = 1 \\ x = 4 \rightarrow u = 10 \end{cases} \quad \text{if we change limits}$$

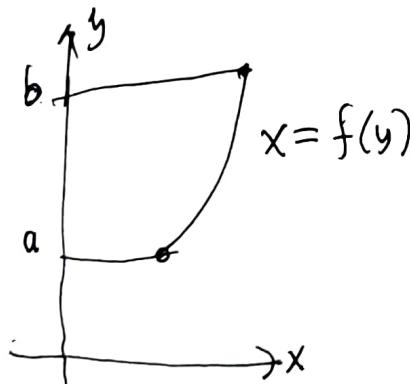
$$\int \sqrt{1 + \frac{9}{4}x} dx = \int u^{1/2} \left(\frac{4}{9}du\right) = \frac{4}{9} \int u^{1/2} du$$

$$= \frac{4}{9} \cdot \frac{u^{3/2}}{3/2} = \frac{8}{27} u^{3/2} = \frac{8}{27} (1 + \frac{9}{4}x)^{3/2}$$

$$L = \frac{8}{27} (1 + \frac{9}{4}x)^{3/2} \Big|_0^4 = \frac{8}{27} \left[(1 + 9)^{3/2} - 1 \right] = \frac{8}{27} (10^{3/2} - 1)$$

 ≈ 9.073

Compare
Secant line length 8.94

example

we can switch x and y
and use the same formula

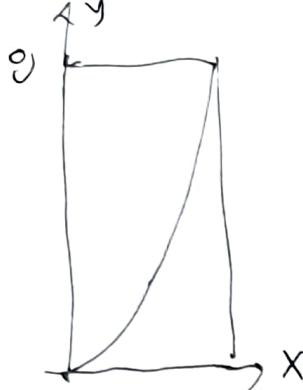
$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2 dy}$$

Why not? Just rotate by 90°
and we are back where we were.

8.1a

arclength of curve segment

(5)

same example again

$$y = x^{3/2} \rightarrow x = y^{2/3}$$

$$\frac{dx}{dy} = \frac{2}{3} y^{-1/3} = \frac{2}{3y^{1/3}}$$

$$1 + \left(\frac{dx}{dy} \right)^2 = 1 + \frac{4}{9} y^{2/3} = \frac{y^{2/3} + 4/9}{y^{2/3}}$$

$$\sqrt{1 + \left(\frac{dx}{dy} \right)^2} = \sqrt{\frac{y^{2/3} + 4/9}{y^{2/3}}} = \frac{\sqrt{y^{2/3} + 4/9}}{y^{1/3}}$$

$$= \sqrt{y^{2/3} + 4/9} y^{-1/3}$$

aha!

$$L = \int_0^8 \sqrt{1 + \left(\frac{dx}{dy} \right)^2} dy$$

$$= \int_0^8 \underbrace{(y^{2/3} + 4/9)}^{u} \underbrace{y^{-1/3} dy}_{\frac{3}{2} du}$$

$$u = y^{2/3} + 4/9$$

$$du = \frac{2}{3} y^{-1/3} dy$$

$$\frac{3}{2} du = y^{-1/3} dy$$

simple u-sub
derivative factor
escaped from sqrt!

$$\int u^{1/2} \left(\frac{3}{2} du \right) = \frac{3}{2} \int u^{1/2} du = \frac{3}{2} \frac{u^{3/2}}{3/2} + C = u^{3/2} + C$$

$$= (y^{2/3} + 4/9)^{3/2}$$

$$L = (y^{2/3} + 4/9)^{3/2} \Big|_0^8 = \underbrace{(8^{2/3} + 4/9)^{3/2}}_{2^2=4} - \underbrace{(4/9)^{3/2}}_{(\frac{2}{3})^3 = \frac{8}{27}}$$

$$4 + \frac{4}{9} = 4 \left(1 + \frac{1}{9} \right)$$

$$= 4 \left(\frac{10}{9} \right)$$

$$\left(\frac{4}{9} \cdot 10 \right)^{3/2} = \frac{8}{27} 10^{3/2}$$

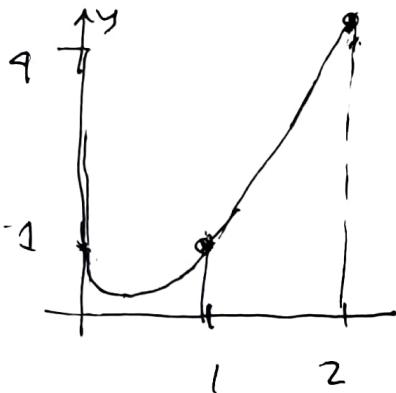
$$= \frac{8}{27} (10^{3/2} - 1) \quad \checkmark$$

B1a) arclength of a curve

(6)

example

almost all elementary calculus examples where a "simple" antiderivative exists are due to a **PERFECT SQUARE** inside the sqrt (if not a simple u-sub because of a factor that factors out of the sqrt like before).



$$y = x^2 - \frac{1}{8} \ln x$$

$\hookrightarrow x > 0$ avoid

$$\frac{dy}{dx} = 2x - \frac{1}{8} \left(\frac{1}{x} \right)$$

$$1 + \left(\frac{dy}{dx} \right)^2 = \left(1 + 4x^2 - \frac{1}{2} \left(x \cdot \frac{1}{x} \right) + \frac{1}{64x^2} \right) = \underbrace{\frac{1}{2}}_{\left(2x + \frac{1}{8x} \right)^2} \text{ perfect square}$$

$$L = \int_1^2 \sqrt{\left(2x + \frac{1}{8x} \right)^2} dx$$

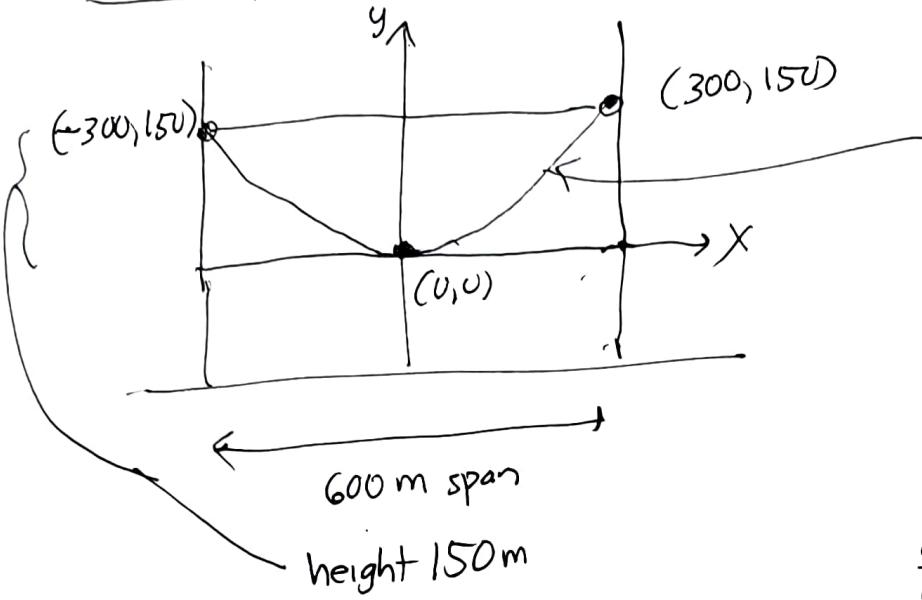
$$= \int_1^2 2x + \frac{1}{8x} dx = x^2 + \frac{1}{8} \ln x \Big|_1^2$$

$$= 2^2 + \frac{1}{8} \ln 2 - 1^2 - \frac{1}{8} \ln 1 = 3 + \frac{1}{8} \ln 2 \approx 3.087$$

8.1.9 arc length of curve segment

(7)

text example bridge cable hanging in parabolic shape



$$\text{shape: } y = ax^2$$

$$150 = a(300)^2$$

$$a = \frac{150}{300^2}$$

$$y = \frac{150}{300} x^2 = 150 \left(\frac{x}{300}\right)^2$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{150}{300^2} (2x) \\ &= \frac{x}{300} \end{aligned}$$

$$L = \int_{-300}^{300} \sqrt{1 + \left(\frac{x}{300}\right)^2} dx = 2 \int_0^{300} \sqrt{1 + \left(\frac{x}{300}\right)^2} dx \quad \text{symmetry!}$$

$$\underset{\substack{\text{Maple} \\ \text{antiderivative}}}{=} 2 \left[\frac{x \sqrt{x^2 + 90,000}}{600} + 150 \operatorname{arcsinh}\left(\frac{x}{300}\right) \right] \Big|_0^{300}$$

$$= 2 \left[\frac{1}{2} \sqrt{90,000 + 90,000} + 150 \operatorname{arcsinh}(1) \right]$$

$$= 300\sqrt{2} + 300 \underbrace{\operatorname{arcsinh}(1)}_{\ln(1+\sqrt{1+1})}$$

$$\begin{aligned} \operatorname{arcsinh}(x) \\ = \ln(x + \sqrt{x^2 + 1}) \end{aligned}$$

$$\approx 688.7 \text{ m}$$

compare to straight lines

$$300\sqrt{5} \approx 670.8$$

must be somewhat bigger, checks out!

