

7.8 c

Indefinite Integrals: comparison and applications ①

When an integrand has no easily expressed antiderivative, we cannot evaluate the necessary limits to [test] for convergence or divergence. Knowing that an integral converges means it makes sense to try to evaluate it numerically. We also need to be able to identify convergence and divergence of integrals over semi-infinite intervals for [infinite series] applications.

The ["comparison theorem"] is intuitively obvious, following from the squeeze theorem on limits.

Consider $\int_a^b f(x) dx$ of either type improper integral

Squeeze the integrand or force it higher.

$$0 \leq f(x) \leq g(x)$$

$$0 \leq g(x) \leq f(x)$$

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx$$

↓

if this converges

then this must converge

$$\int_a^b g(x) dx \leq \int_a^b f(x) dx$$

if this diverges

then this must diverge

example

$$\int_2^\infty \frac{1}{\sqrt{x}-1} dx$$

$\underbrace{\quad}_{> \frac{1}{\sqrt{x}} > 0}$

denom smaller so fraction bigger

$$> \int_2^\infty x^{-1/2} dx = \lim_{t \rightarrow \infty} \int_2^t x^{-1/2} dx$$

$$= \lim_{t \rightarrow \infty} \left. \frac{x^{1/2}}{1/2} \right|_2^t$$

$$= \lim_{t \rightarrow \infty} (2t^{1/2} - 2 \cdot 2^{1/2}) \underset{\downarrow \infty}{\text{diverges}} = \infty$$

so original integral diverges

example

$$\int_0^\infty \frac{\arctan x}{2+e^x} dx \quad 0 < \frac{\arctan x}{2+e^x} < \frac{\frac{\pi}{2}}{2+e^x} < \frac{\pi/2}{e^x} = \frac{\pi}{2} e^{-x}$$

denom smaller
fraction larger

$$< \int_0^\infty \frac{\pi}{2} e^{-x} dx = \lim_{t \rightarrow \infty} \frac{\pi}{2} \int_0^t e^{-x} dx = \lim_{t \rightarrow \infty} \frac{\pi}{2} \frac{e^{-x}}{-1} \Big|_0^t$$

$$= \lim_{t \rightarrow \infty} -\frac{\pi}{2} (e^{-t} - 1) = \frac{\pi}{2} \text{ converges}$$

so original converges

Maple: $\approx 0.41 < 1.57$

$$\text{example} \quad \int_0^\infty \frac{1}{1+x^3} dx \quad \frac{1}{1+x^3} < \frac{1}{0+x^3} = x^{-3} \text{ but division by } 0 \text{ at } x=0 \text{ so}$$

$$\underbrace{\int_0^1 \frac{1}{1+x^3} dx}_{\text{finite}} + \underbrace{\int_1^\infty \frac{1}{1+x^3} dx}_{< \int_1^\infty x^{-3} dx} = \lim_{t \rightarrow \infty} \int_1^t x^{-3} dx$$

$$= \lim_{t \rightarrow \infty} \frac{x^{-2}}{-2} \Big|_1^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{2t^2} + \frac{1}{2} \right) \rightarrow 0$$

$= \frac{1}{2}$ converges so
original converges

Maple: $= \frac{3\pi\sqrt{3}}{9} \approx 1.21$

7.8c

Improper Integrals: comparison and applications

(3)

Remark In fact only the behavior "near infinity" or "near a vertical asymptote matters."

example $\int_0^\infty \frac{2x^2+1}{3x^4+x+1} dx = \int_0^{10^6} \frac{2x^2+1}{3x^4+x+1} dx + \int_{10^6}^\infty \frac{2x^2+1}{3x^4+x+1} dx$

finite test this

$\approx \frac{2x^2}{3x^4} = \frac{2}{3}x^{-2}$ ← as long as adding 1 to the negative power is still negative, the contribution from $x \rightarrow \infty$ vanishes

as $x \rightarrow \infty$ integrate

$\frac{2}{3}x^{-1} \rightarrow 0$ as $x \rightarrow \infty$ so converges

Maple:
 ≈ 1.52

OR

example $\int_0^1 \frac{e^{x^2}}{x^{1/2}} dx$ ($\left(< \int_0^1 \frac{e^x}{x^{1/2}} dx = 2e \approx 5.44 \right)$)

$\approx \frac{1}{x^{1/2}}$ as $x \rightarrow 0^+$, $e^{x^2} \rightarrow e^0 = 1$

so the integrand looks more and more like

$\frac{1}{x^{1/2}} \xrightarrow{\text{integrate}} \frac{x^{1/2}}{1/2} \rightarrow 0$ as $x \rightarrow 0$ so no division by zero at the origin as in the integrand.

Maple:
 ≈ 2.54

converges.

7.8c)

Improper Integrals: comparison and applications

(4)

Improper integrals are not just mathematical curiosities, they are fundamental to many applications, both mathematical like the normal distribution as well as in STEM fields.

[problem 7.8.62] is a great example.

An "ideal gas" is a distribution of molecules moving at different speeds determined by its temperature.

parameters:

M molecular weight

T temperature (°K)

R gas constant

v molecular speed

Average speed formula:
 shorthand $\bar{v} = \frac{4}{\sqrt{\pi}} \left(\frac{M}{2RT} \right)^{3/2} \int_0^{\infty} v^3 e^{-\frac{mv^2}{2RT}} dv$

How to make sense of this mess? Group factors first:

$$\bar{v} = \frac{4}{\sqrt{\pi}} \int_0^{\infty} \left(\frac{v}{\sqrt{2RT/M}} \right)^3 e^{-\left(\frac{v}{\sqrt{2RT/M}} \right)^2} dv$$

must be dimensionless so



$$v = \sqrt{\frac{2RT}{M}}$$
 has velocity dimensions

introduce dimensionless speed

$$u = \frac{v}{\bar{v}}, du = \frac{dv}{\bar{v}}, dv = \bar{v} du$$

$$v=0 \rightarrow u=0$$

$$v \rightarrow \infty \rightarrow u \rightarrow \infty$$

same limits

$$\bar{v} = \frac{4}{\sqrt{\pi}} \int_0^{\infty} u^3 e^{-u^2} (\bar{v} du)$$

$$= \bar{v} \int_0^{\infty} u P(u) du$$

average value of

u is a weighted average of u weighted by P(u)

$$P(u) = \frac{4}{\sqrt{\pi}} u^2 e^{-u^2}$$

$$\int_0^{\infty} P(u) du = 1 \quad \text{probability distribution!}$$

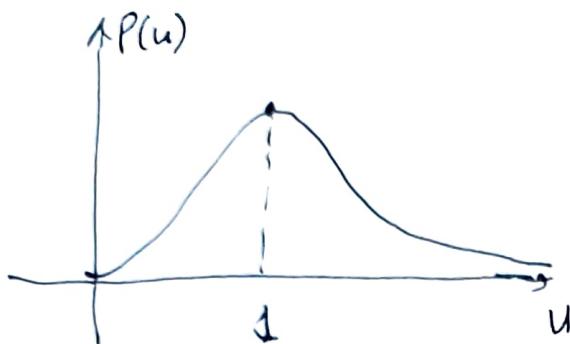
(stewart)
(8.5!)

7.8c

Improper Integrals: comparison and applications

(5)

What does this look like?



most of molecules move near speed V

$$\begin{aligned}
 \text{peak: } D &= P'(u) = \frac{d}{du} (u^2 e^{-u^2}) \\
 &= 2ue^{-u^2} + u^2(e^{-u^2})(-2u) \\
 &= 2ue^{-u^2}(1-u^2) \\
 &\hookdownarrow u=1 \\
 &\downarrow v=V
 \end{aligned}$$

peaked at speed V .

but average value is slightly to the right

$$\bar{v} = V \int_0^\infty \frac{4}{\sqrt{\pi}} u^3 e^{-u^2} du = \frac{2}{\sqrt{\pi}} V \approx 1,129 \text{ V}$$

↓ just a number ↑

dimensional analysis gives final result modulo a numerical factor (change of variable!) nearly equal to 1.

Details:

$$\begin{aligned}
 \int u^3 e^{-u^2} du &= \int \underbrace{u^2}_{w=-u^2} e^{-\underbrace{u^2}_w} \underbrace{u du}_{-\frac{1}{2} dw} = \frac{1}{2} \int w e^w dw \\
 &= \frac{1}{2} (w-1) e^w + C \\
 &= -\frac{1}{2} (u^2+1) e^{-u^2} + C
 \end{aligned}$$

$$\begin{aligned}
 \int_0^\infty u^3 e^{-u^2} du &= \lim_{t \rightarrow \infty} \int_0^t u^3 e^{-u^2} du = \lim_{t \rightarrow \infty} \left[-\frac{1}{2} (u^2+1) e^{-u^2} \right]_0^t \\
 &= \lim_{t \rightarrow \infty} \left(-\frac{1}{2} \cancel{\frac{t^2+1}{e^{t^2}}} + \frac{1}{2} (1)(1) \right) = \frac{1}{2}
 \end{aligned}$$

$$\frac{4}{\sqrt{\pi}} \left(\frac{1}{2} \right) = \frac{2}{\sqrt{\pi}} \quad V$$