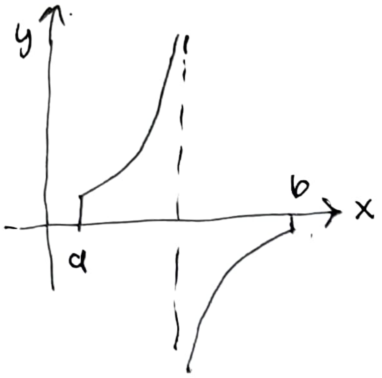
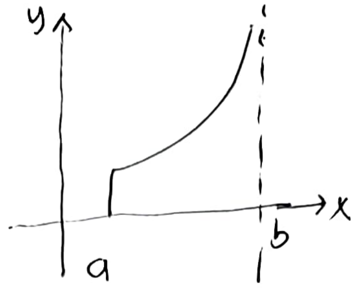
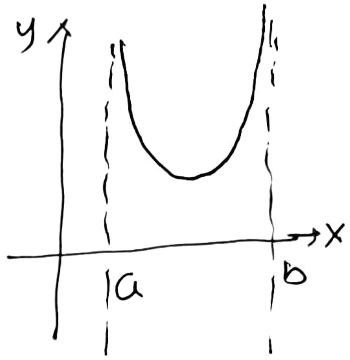
$\boxed{7.8b}$ Improper Integrals: implicit infinities

Improper integrals over a closed interval $a \le x \le b$ where $y \to \pm\infty$ for $y = f(x)$ are of 3 types:





at one endpoint

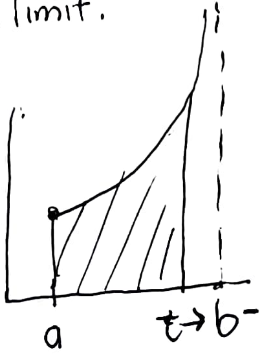


at both endpoints

$\boxed{\text{vertical asymptote}}$ at 1 or more pts <u>inside</u> the interval

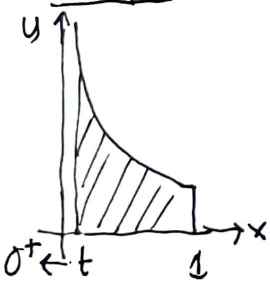
The key idea is to divide up the interval into subintervals involving at most one infinity, and approach the single location of a vertical asymptote with a one-sided limit.

$\boxed{\text{one endpoint asymptote}}$



$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

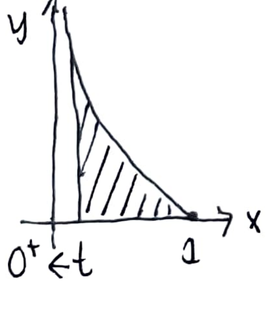
<u>example</u>



$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-1/2}\,dx = \lim_{t\to 0^+} \frac{x^{1/2}}{1/2}\Big|_t^1$$

$$= \lim_{t\to 0^+} 2(1-\underbrace{t^{1/2}}_{\to 0!}) = 2 \quad \text{converges}$$

<u>example</u>



$$\int_0^1 -\ln x\,dx = \lim_{t\to 0^+} \int_t^1 -\ln x\,dx = \lim_{t\to 0^+} x(1-\ln x)\Big|_t^1$$

$$= \lim_{t\to 0^+} \underbrace{1(1-\underbrace{\ln 1}_{\to 0})}_{1} - t(1-\ln t) = \lim_{t\to 0^+} (1 - \underbrace{t}_{0} + \underbrace{t\ln t}_{0\,\cdot\,\infty})$$

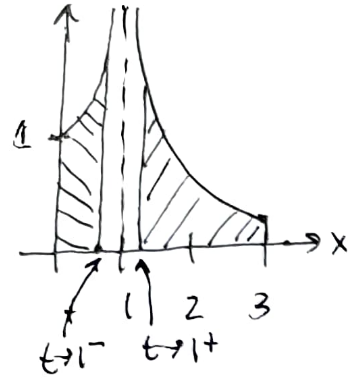
l'Hopital: $\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{t^{-1}} = \lim_{t\to 0^+} \frac{1/t}{-t^{-2}} = \lim_{t\to 0^+} -t = 0$

" $\frac{\infty}{\infty}$ "

$= 1$ converges

7.8b Improper Integrals: implicit infinities

asymptote inside interval



$$\int_0^3 \underbrace{\frac{1}{(x-1)^2}}_{>0}\,dx \neq -\frac{1}{x-1}\Big|_0^3 = -\frac{1}{2}-1=-\frac{3}{2}<0\ !!$$

fund theorem of calc Does Not Apply!

$$= \int_0^1 \frac{dx}{(x-1)^2} + \int_1^3 \frac{dx}{(x-1)^2}$$

$$= \lim_{t\to 1^-}\int_0^t (x-1)^{-2}\,dx + \lim_{t\to 1^+}\int_t^3 (x-1)^{-2}\,dx$$

$$= \lim_{t\to 1^-} \frac{1}{1-x}\Big|_0^t + \lim_{t\to 1^+} \frac{1}{1-x}\Big|_t^3$$

$$= \underbrace{\lim_{t\to 1^-}\left(\underbrace{\frac{1}{1-t}}_{\to+\infty}-1\right)}_{\infty} + \underbrace{\lim_{t\to 1^+}\left(\frac{1}{-2}-\underbrace{\frac{1}{1-t}}_{\to+\infty}\right)}_{\infty} = \infty$$

diverges

Remark

$$\int \frac{1}{(x-a)^p}\,dx = \int (x-a)^{-p}\,dx \underset{p\neq 1}{=} \frac{(x-a)^{1-p}}{1-p}$$

$p>0$ for division by 0

$1-p>0$ for NO division by 0

$\downarrow$

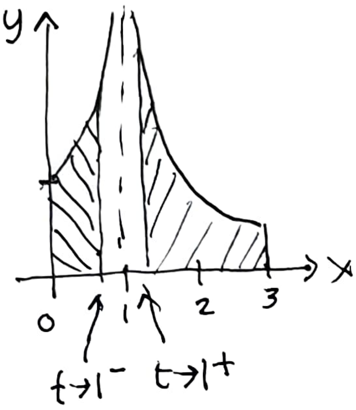
$1>p$

$\boxed{0<p<1}$

proper fractions

Improper Integrals: implicit infinities

example



$$\int_0^3 \frac{1}{\underbrace{\sqrt{|x-1|}}_{\text{piecewise defined function}}}\,dx = \int_0^1 \frac{1}{\sqrt{1-x}}\,dx + \int_1^3 \frac{1}{\sqrt{x-1}}\,dx$$

$$= \lim_{t\to 1^-} \int_0^t (1-x)^{-1/2}\,dx + \lim_{t\to 1^+} \int_t^3 (x-1)^{-1/2}\,dx$$

$$= \lim_{t\to 1^-} -\frac{(1-x)^{1/2}}{1/2}\Bigg|_0^t + \lim_{t\to 1^+} +\frac{(x-1)^{1/2}}{1/2}\Bigg|_t^3$$

$$= \lim_{t\to 1^-} -2\underbrace{(1-t)^{1/2}}_{\to 0} + \underbrace{2(1-0)^{1/2}}_{2} + \lim_{t\to 1^+} \underbrace{2(3-1)^{1/2}}_{2\sqrt{2}} - 2\underbrace{(t-1)^{1/2}}_{\to 0}$$

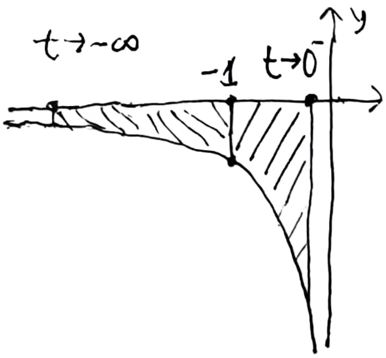
$$= 2 + 2\sqrt{2}$$ converges

Improper Integrals: implicit infinities

example 7.8.39 extended combo of explicit/implicit

$$\int_{-\infty}^{0} \underbrace{\frac{e^{\frac{1}{x}}}{x^3}}_{<0} dx = \int_{-\infty}^{-1} \frac{e^{1/x}}{x^3}\,dx + \int_{-1}^{0} \frac{e^{1/x}}{x^3}\,dx$$

$\begin{pmatrix} x \to -\infty \\ e^{1/x} \to e^0 = 1 \end{pmatrix}$ $\quad (x \to 0^-,\ e^{1/x} \to e^{-\infty} = 0)$



$$= \lim_{t\to-\infty} \int_t^{-1} \frac{e^{1/x}}{x^3}\,dx + \lim_{t\to 0^-} \int_{-1}^{0} \frac{e^{1/x}}{x^3}\,dx$$

antiderivative:

$$\int \frac{e^{1/x}}{x^3}\,dx = \int \frac{1}{x} e^{1/x} \frac{dx}{x^2} = \int u e^u (-du)$$

$u = 1/x$

$du = -x^{-2}dx$

$$= -(u-1)e^u + C$$
$$= (1-u)e^u + C$$
$$= \left(1-\frac{1}{x}\right)e^{1/x} + C$$

$$= \lim_{t\to-\infty} \left(1-\frac{1}{x}\right)e^{1/x}\Big|_t^{-1} + \lim_{t\to 0^-} \left(1-\frac{1}{x}\right)e^{1/x}\Big|_{-1}^{t}$$

$$= \underbrace{\lim_{t\to-\infty} \underbrace{(1-(-1))}_{2}e^{-1} - \underbrace{\left(1-\frac{1}{t}\right)e^{1/t}}_{1}}_{= 2e^{-1} - 1 < 0}$$

converges

$$+ \lim_{t\to 0^-} \underbrace{\underbrace{\left(1-\frac{1}{t}\right)}_{\to+\infty} e^{1/t}}_{\ } - \underbrace{(1-(-1))}_{2} e^{-1}, \quad e^{1/t} \to e^{-\infty} = 0$$

$$\underbrace{\lim_{t\to 0^-} \frac{1-t^{-1}}{e^{-t^{-1}}} \ \text{"}\frac{\infty}{\infty}\text{"} = \lim_{t\to 0^-} \frac{\cancel{t^{-2}}}{e^{-t^{-1}}\cancel{t^{-2}}} = \lim_{t\to 0^-} e^{\frac{1}{t}} \to e^{-\infty} = 0}_{-2e^{-1}}$$

converges

$$= 2e^{-1} - 1 - 2e^{-1} = -1 < 0 \ \checkmark$$

converges

alternative approach: change limits in variable change

$$\int_{-\infty}^{-1} \underbrace{\frac{e^{\frac{1}{x}}}{x^3}\,dx}_{-ue^u\,du} = \underbrace{\int_0^{-1} -ue^u\,du}_{\text{proper integral!}} = (1-u)e^u\Big|_0^{-1}$$

$= (+1)e^{-1} - 1e^0$

$= 2e^{-1} - 1$

$u = \frac{1}{x}$

$x \to -\infty,\ u \to 0^-$

$x = -1,\ u = -1$

$$\int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^3}\,dy = \int_{-1}^{-\infty} -ue^u\,du = \lim_{t\to-\infty} (1-u)e^u\Big|_{-1}^{t}$$

$x \to 0^-,\ u \to -\infty$

$x = -1,\ u = -1$

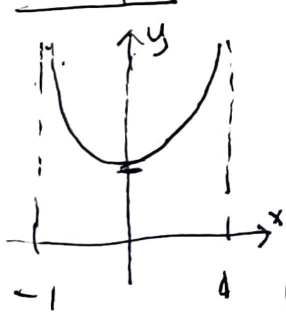
$$= \lim_{t\to-\infty}[\underbrace{(1-t)e^t}\ \underbrace{-(1+1)e^{-1}}_{-2e^{-1}}]$$

$$\lim_{t\to-\infty} \frac{1-t}{e^{-t}} = \lim_{t\to-\infty} \frac{-1}{e^{-t}(-1)} = \lim_{t\to-\infty} e^{t} = 0$$

$\text{"}\frac{\infty}{\infty}\text{"}$

$= -2e^{-1}$

example



$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \underbrace{\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx}_{\text{same by symmetry.}} + \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{t\to1^-} \int_0^t (1-x^2)^{-1/2}\,dx$$

$$= \lim_{t\to1^-} \arcsin x\Big|_0^t$$

$$= \lim_{t\to1^-} \arcsin 1 - \arcsin 0 = \arcsin 1 = \frac{\pi}{2}$$

$= 2\left(\frac{\pi}{2}\right)$

$= \pi$

converges