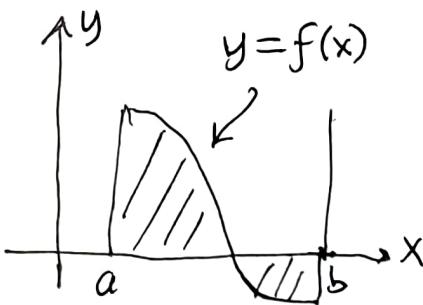


7.84

Improper Integrals

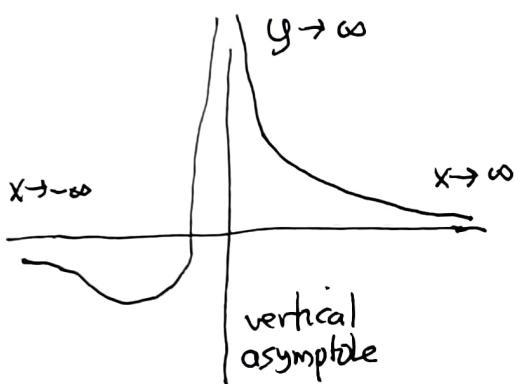
①

"Proper integrals"

$$\int_a^b f(x) dx = F(x) \Big|_a^b$$

continuous (or at least  
piecewise continuous)  
on  $a \leq x \leq b$

"improper integrals" involve infinity either explicitly in the limits of integration or implicitly in the values of  $f(x)$  in the interval of integration or both



1) type 1:  $x \rightarrow \pm\infty$  in the sense:  
 $a \rightarrow -\infty$  or  $b \rightarrow \infty$  or both

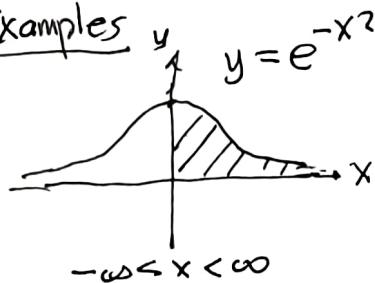
2) type 2:  $y = f(x) \rightarrow \pm\infty$  at some value of  $x$  within the interval of integration.

To make sense of these integrals, each infinity which occurs must be made explicit as an infinity and if more than one occurs in an integral, we must do so separately by dividing up the interval of integration.

7.8a

Improper Integrals

(2)

Examples

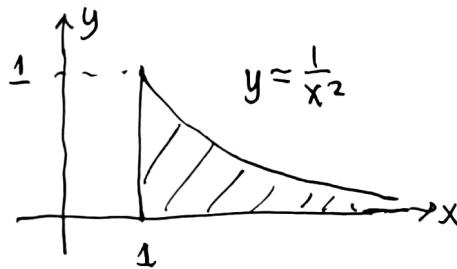
bell curve—  
normal distribution  
in probability.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = ?$$

2 infinities.

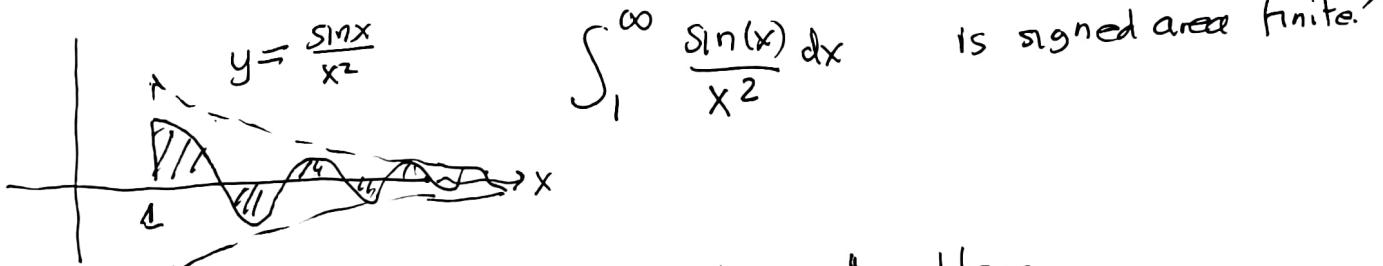
$$\int_0^{\infty} e^{-x^2} dx = ?$$

1 infinity  
Is either area finite?

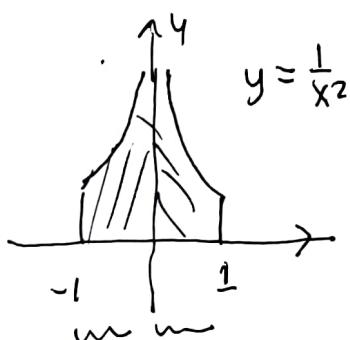


$$\int_1^{\infty} \frac{1}{x^2} dx = ?$$

Is area finite.



all "type 1" problems.



either can be infinite  
Separately and we  
cannot add/subtract  
infinities

$$\int_{-1}^1 \frac{1}{x^2} dx = ?$$

$y \rightarrow \infty$  when  $x \rightarrow 0$   
vertical asymptote

$$\int_0^1 \frac{1}{x^2} dx = ?$$

now at endpoint.

$y \rightarrow \infty$

$$\int_{-1}^1 \frac{1}{x^2} dx = \int_{-1}^0 \frac{1}{x^2} dx + \int_0^1 \frac{1}{x^2} dx$$

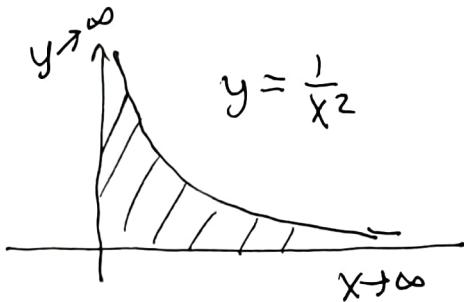
$y \rightarrow \infty$

must analyze each separately  
(ignoring symmetry)  
here!

7.8-a

Improper Integrals

(3)



divide and conquer

$$\int_0^\infty \frac{1}{x^2} dx = ?$$

$$= \int_0^1 \frac{1}{x^2} dx + \int_1^\infty \frac{1}{x^2} dx$$

$y \rightarrow 0$  only one infinity per interval  
(1 arbitrary!)

Today we consider only explicit infinite limits of integration.  
Then implicit infinities and finally combinations.

vocabulary of limits

When the limit of an expression is finite we say it  
converges

When the limit of an expression is either infinite or  
does not exist, we say it

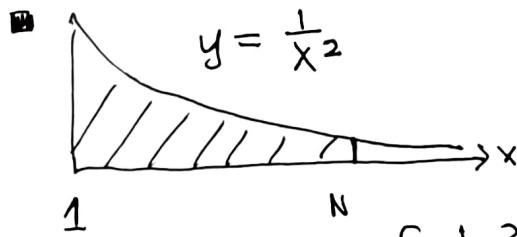
diverges

We use the same terms for improper integrals, all of  
which must be re-expressed explicitly as limits.

## 7.8a) Improper Integrals

(4)

### power functions



Is area under curve finite?

Define as a limit:

$$\int_1^\infty x^{-2} dx = \lim_{N \rightarrow \infty} \int_1^N x^{-2} dx$$

$$= -x^{-1} \Big|_1^N = -\frac{1}{N} + 1$$

$$= \lim_{N \rightarrow \infty} \left( 1 - \frac{1}{N} \right) = 1 \quad \text{converges } \checkmark$$

similarly (same picture):

$$\int_1^\infty \frac{1}{x} dx = \lim_{N \rightarrow \infty} \int_1^N x^{-1} dx = \lim_{N \rightarrow \infty} \ln N = \infty \quad \text{diverges } \checkmark$$

$$\ln x \Big|_1^N = \ln N - \ln 1$$

other positive powers of  $x$  in the denominator?  
(otherwise no hope!)

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{N \rightarrow \infty} \int_1^N x^{-p} dx = \lim_{N \rightarrow \infty} \frac{x^{1-p}}{1-p} \Big|_1^N$$

$\uparrow$   
 $(p > 0 \text{ but } p \neq 1)$

$$= \lim_{N \rightarrow \infty} \frac{\frac{N^{1-p}}{1-p} - \frac{1}{1-p}}{\frac{1}{1-p}} = \frac{1}{p-1} > 0$$

$\nearrow$   
if  $1-p < 0$   
 $1 < p$   
 $p > 1$

converges  $\checkmark$

and diverges for  $0 < p < 1$

7.8a

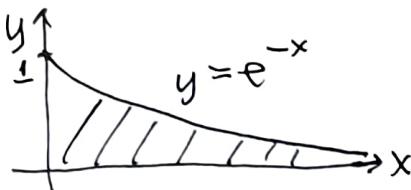
Improper integrals

(5)

exponential functions

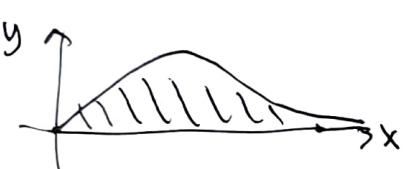
■  $\int_0^\infty e^{-x} dx = \lim_{N \rightarrow \infty} \int_0^N e^{-x} dx = \lim_{N \rightarrow \infty} \left[ -e^{-x} \right]_0^N = \lim_{N \rightarrow \infty} (-e^{-N} + 1) = 1$

converges ✓



■  $\int_0^\infty x e^{-x} dx = \lim_{N \rightarrow \infty} \int_0^N x e^{-x} dx = \lim_{N \rightarrow \infty} \left[ (x-1)e^{-x} \right]_0^N = \lim_{N \rightarrow \infty} (N-1)e^{-N} - (-1) = 0$

exponentials beat  
any polynomials  
(L'Hopital's rule)



■  $\int_0^\infty \underbrace{\text{Poly}(x)}_{\text{antiderivative}} e^{-x} dx = \dots$  converges

$\frac{Q(x)}{e^x}$



7.8a

Improper Integrals

(6)

$$\int_{-\infty}^{\infty} \frac{1+x}{1+x^2} dx = \int_{-\infty}^0 \frac{1+x}{1+x^2} dx + \int_0^{\infty} \frac{1+x}{1+x^2} dx$$

↑      ↓

$$= \lim_{N \rightarrow -\infty} \int_N^0 \frac{1+x}{1+x^2} dx + \underbrace{\lim_{N \rightarrow \infty} \int_0^N \frac{1+x}{1+x^2} dx}$$

can only add limits if both finite or  
assign infinite value if both have same sign.

$$\int \frac{1+x}{1+x^2} dx = \underbrace{\int \frac{1}{1+x^2} dx}_{\arctan x} + \frac{1}{2} \underbrace{\int \frac{2x}{1+x^2} dx}_{\int \frac{du}{u} = \ln u}$$

$$= \arctan x + \frac{1}{2} \ln(1+x^2) + C$$

$$\int_{-\infty}^0 \frac{1+x}{1+x^2} dx = \lim_{N \rightarrow -\infty} \left( \arctan x + \frac{1}{2} \ln(1+x^2) \right) \Big|_0^N = \lim_{N \rightarrow \infty} \left( -\arctan N - \frac{1}{2} \ln(1+N^2) \right) \Big|_{-\frac{\pi}{2}}^{-\infty}$$

$$0 + \frac{1}{2} \lim_{N \rightarrow 0} (-\arctan N - \frac{1}{2} \ln(1+N^2))$$

$$= -\infty \text{ diverges}$$

$$\int_0^{\infty} \frac{1+x}{1+x^2} dx = \lim_{N \rightarrow \infty} \left( \arctan x + \frac{1}{2} \ln(1+x^2) \right) \Big|_0^N = \lim_{N \rightarrow \infty} \left( \arctan x + \frac{1}{2} \ln(1+N^2) \right) \Big|_{\frac{\pi}{2}}^{+\infty}$$

$$\arctan N + \frac{1}{2} \ln(1+N^2) - 0 - \frac{1}{2} \ln 0$$

$= \infty$  diverges. and  $\int_{-\infty}^{\infty} \frac{1+x}{1+x^2} dx$  does not exist!  
diverges

But

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = 2 \int_0^{\infty} \frac{dx}{1+x^2} = 2 \lim_{N \rightarrow \infty} \int_0^N \frac{dx}{1+x^2} = 2 \lim_{N \rightarrow \infty} \arctan x \Big|_0^N$$

$$= 2 \lim_{N \rightarrow \infty} (\arctan N - 0) = 2 \left( \frac{\pi}{2} \right) = \pi$$

converges!