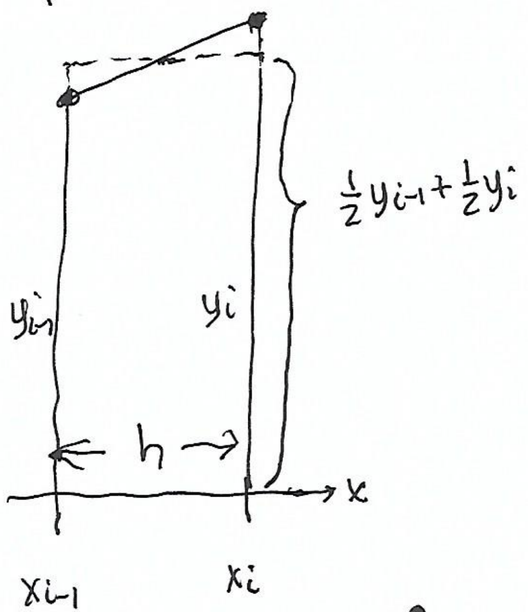
Numerical Integration: trapezoid versus Simpson rules

trapezoid rule: $y = f(x)$, $y_i = f(x_i)$



$$\Delta A_i = \underbrace{h}_{\text{width}} \underbrace{\left(\tfrac{1}{2} y_{i-1} + \tfrac{1}{2} y_i\right)}_{\text{height of equivalent rectangle}}$$

consider a pair of subintervals to compare to Simpson.

$$\Delta A_{i-1} + \Delta A_i$$

$$= h\left(\tfrac{1}{2} y_{i-2} + \tfrac{1}{2} y_{i-1}\right) + h\left(\tfrac{1}{2} y_{i-1} + \tfrac{1}{2} y_i\right)$$

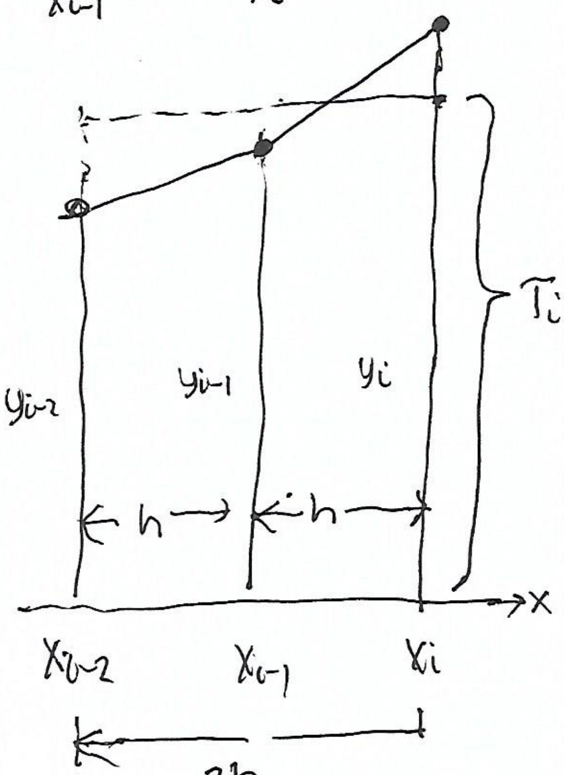
$$= h\left(\tfrac{1}{2} y_{i-2} + y_{i-1} + \tfrac{1}{2} y_i\right)$$

$$= \underbrace{2h}_{\text{width}} \underbrace{\left(\tfrac{1}{4} y_{i-2} + \tfrac{2}{4} y_{i-1} + \tfrac{1}{4} y_i\right)}_{\text{height of equivalent rectangle } T_i}$$

= weighted average of 3 endpoint values

gives twice weight to midpoint value, the endpoints share half the remaining weight

(makes sense to give more weight to midpoint as we already saw)



Compare $\Delta A_{i-1} + \Delta A_i = \begin{cases} 2h\left(\tfrac{1}{4} y_{i-2} + \tfrac{2}{4} y_{i-1} + \tfrac{1}{4} y_i\right) & \text{trapezoid} \\ \quad\Updownarrow \qquad\quad \Updownarrow \qquad\quad \Updownarrow & \\ 2h\left(\tfrac{1}{6} y_{i-2} + \tfrac{4}{6} y_{i-1} + \tfrac{1}{6} y_i\right) & \text{simpson} \end{cases}$

more weight to midpoint leads to much better area approximation

We return to "weighted averages" in section 8.5.

7.7 Approximate Integration Error

$$\int_0^1 \underbrace{e^{x^2}}\,dx \underset{\text{Maple}}{\approx} 1.462651746$$

How does Maple know this is accurate to 10 digits?

no antiderivative expressible in terms of "elementary" functions

A numerical analysis course explores exactly these kinds of questions (advanced mathematics). Here one finds a theoretical application of integration by parts in deriving estimates for the error made in the trapezoid, midpoint and Simpson rule approximations for example.

$\int_a^b f(x)\,dx$ with n equal divisions $\Delta x = \frac{b-a}{n}$

Trapezoid: $|\text{Error}_T| \le \dfrac{K_T(b-a)^3}{12\,n^2} = \dfrac{K_T(b-a)}{12}\underbrace{\left(\dfrac{b-a}{n}\right)^2}_{(\Delta x)^2}$ $\quad K_T = K_M = \max|f''(x)|$ on $[a,b]$

Midpoint: $|\text{Error}_M| \le \dfrac{K_M(b-a)^3}{24\,n^2} = \dfrac{K_M(b-a)}{24}\underbrace{\left(\dfrac{b-a}{n}\right)^2}_{(\Delta x)^2}$

Simpson: $|\text{Error}_S| \le \dfrac{K_S(b-a)^5}{180\,n^4} = \underbrace{\dfrac{K_S(b-a)}{180}}_{\text{fixed}}\underbrace{\left(\dfrac{b-a}{n}\right)^4}_{(\Delta x)^4}$ $\quad K_S = \max|f^{(4)}(x)|$ on $[a,b]$

For the trapezoid and midpoint rules the error goes down like the square of the stepsize Δx but like the fourth power for Simpson.

The fourth power gets smaller MUCH quicker than the square, explaining why Simpson is so much more efficient.

However, these are upper bounds on the error BUT the actual errors may be much less if you compare to the "exact values."