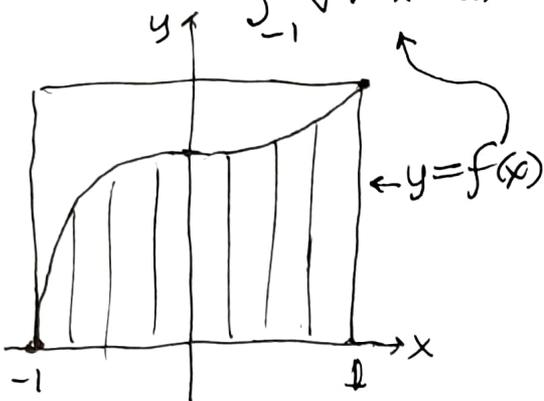


7.7 Approximate Integration

(1)

Example: $\int_{-1}^1 \sqrt{1+x^3} dx$



Even Maple resorts to a complicated complex number formula involving a special function (Elliptic) for the antiderivative.

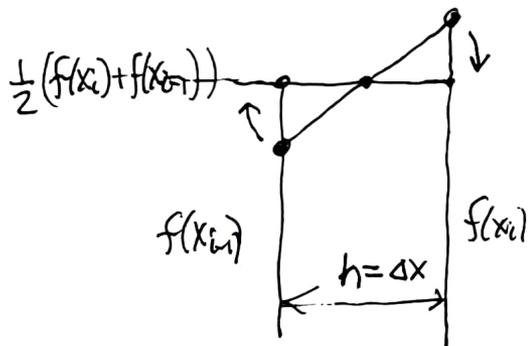
How were these kinds of integrals approximated in the old days?

In fact we need better ways of approximating tabular or graphical data functions.

The key is how we approximate the "area" under the graph (in general signed area) on each small subinterval of a Riemann sum. $A \approx \sum_{i=1}^n \Delta A_i$

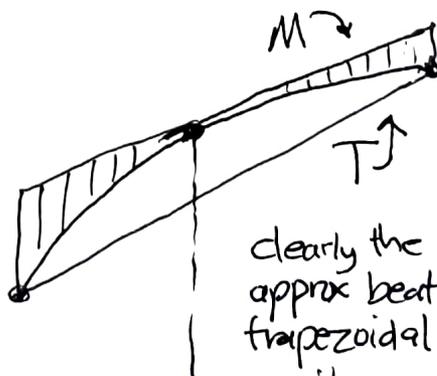
ΔA_i : $f(x_i)$, $f(x_{i-1})$, $f(x_{i-\frac{1}{2}\Delta x})$ + $\frac{1}{2}(f(x_i) + f(x_{i-1}))$
 right left midpoint trapezoid
 R L M " $T = \frac{1}{2}(R+L)$ "

Averaging the left & right Riemann sums is equivalent to using the secant line approximation for the approximating area.



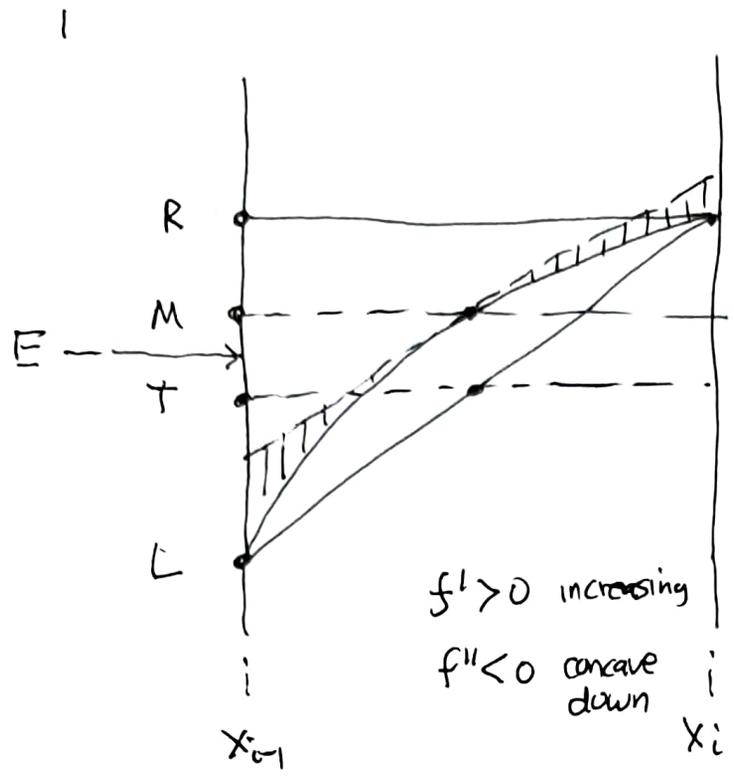
If we rotate the secant line horizontal we get the equivalent rectangle height to compare to the endpoint value approximations.

To compare with midpoint approximation, rotate the horizontal top rectangle edge to the tangent line for an equivalent trapezoidal area



clearly the midpoint approx beats the trapezoidal rule easily

7.7 Approximate Integration



The exact area is in between the midpoint tangent line and the secant line and clearly squeezes closer to the actual area, which in this case is slightly less than the midpoint value

$$L < T < E < M < R$$

We can repeat this diagram for all four cases combining increasing/decreasing with concave up/down. Midpoint always wins. But these are all linear ("straight line") approximations.

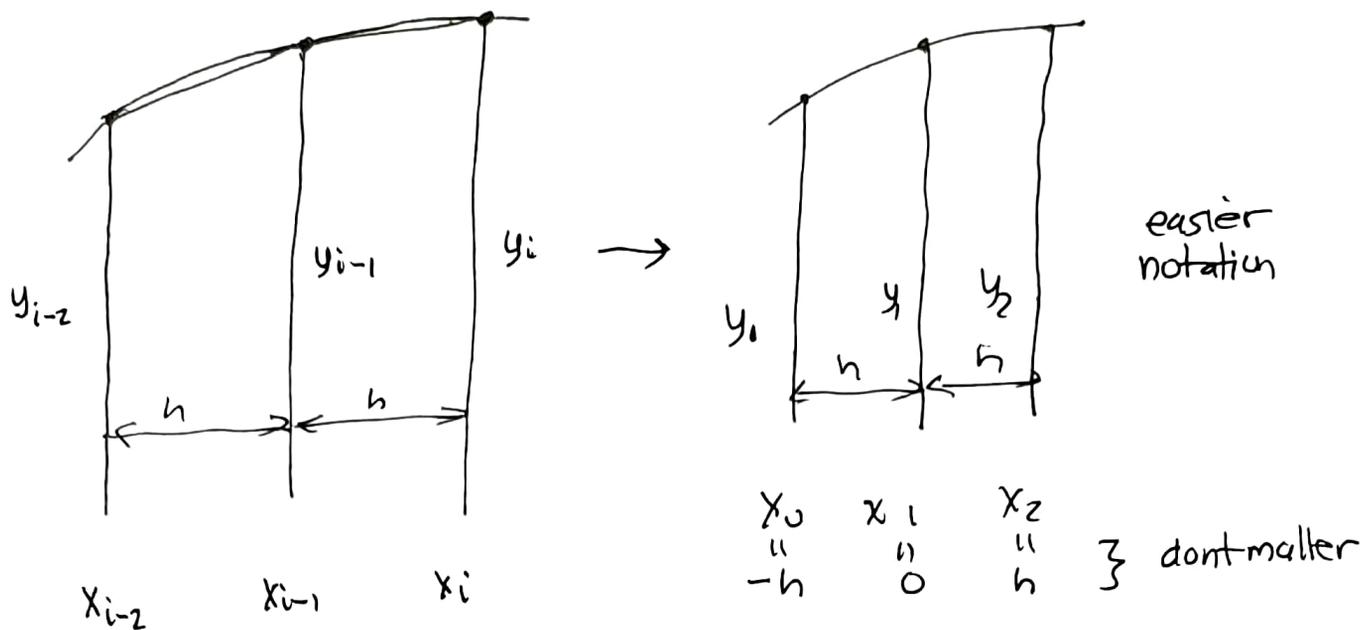
To "fit" a curve, we really need a nonstraight curve. Two points determine a linear function (straight line). Three points determine a quadratic function (parabola).

Simpson's rule does the latter, namely, uses a quadratic approximation.

Of course we can go on and do much better, which is studied in a numerical analysis class, but for dealing with tabular data integration we need a better method which uses all the data points & not just every other one for the midpoint rule, even averaging the left & right endpoint approximations is equivalent to the trapezoidal rule — (midpoint beats trap but half as many pts!) so we really need Simpson for tabular data.

7.7 Approximate Integration

(3)



Divide $x_0=a \dots x_n=b$ into an even number n of subintervals, corresponding to $n+1$ endpoint data points (x_i, y_i) , $i=0, \dots, n$.

Fit 3 data points to a parabola:

$$y = Q(x) = ax^2 + bx + c$$

$$\begin{cases} Q(-h) = a(-h)^2 + b(-h) + c = y_0 \\ Q(0) = a(0)^2 + b(0) + c = y_1 \rightarrow y_1 = c \\ Q(h) = a(h)^2 + b(h) + c = y_2 \end{cases}$$

$$\begin{aligned} \text{sum: } 2ah^2 + 2c &= y_0 + y_2 \rightarrow a = \frac{y_0 - 2y_1 + y_2}{2h^2} \\ \text{difference: } 2bh &= y_2 - y_0 \rightarrow b = \frac{y_2 - y_0}{2h} \end{aligned}$$

back sub into $Q(x)$ and integrate:

$$\begin{aligned} \Delta A_{i,i+1} &= \int_{-h}^h Q(x) dx = \int_{-h}^h (ax^2 + bx + c) dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx \Big|_{-h}^h \\ &= \dots = \frac{h}{3} (y_0 + 4y_1 + y_2) = 2h \left(\frac{1}{6} y_0 + \frac{2}{3} y_1 + \frac{1}{6} y_2 \right) \end{aligned}$$

compare with midpoint rule for double interval $= 2h (0y_0 + 1y_1 + 0y_2)$

The midpoint weakness is that it doesn't take the endpoint behavior into account. Simpson lessens the midpoint a bit and adds in half a contribution from each endpoint to give a much better approximation.

$$\left(\frac{1}{6} + \frac{2}{3} + \frac{1}{6} = 0 + 1 + 0 \right)$$

7.7 Approximate Integration

4

Now add up all these pairs of contributions

$$\begin{aligned} A &\approx \frac{h}{3} (y_0 + 4y_1 + y_2) \\ &\quad + \frac{h}{3} (y_2 + 4y_3 + y_4) \\ &\quad + \frac{h}{3} (y_4 + 4y_5 + y_6) \\ &\quad + \dots \\ &= \frac{h}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \dots + 4y_{n-1} + y_n) \end{aligned}$$

alternates 1, 2, 1, 2, 1 inside
1 on the endpoints.

This is Simpson's rule.

For tabular data or graphical data (estimating tabular data), let Maple sum up these values.

For continuous functions use the Approximate Integration Tutor.

To simply get a highly accurate numerical value, let Maple do it.