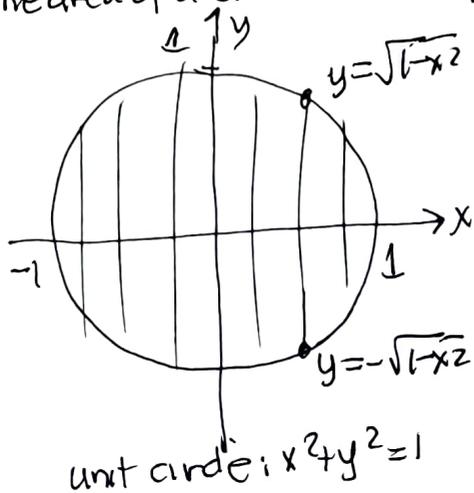


7.1 Integration by parts etc

(1)

Example of what we are not doing

The area of a circle is a simple integral we cannot do.



$$A = \int_{-1}^1 2\sqrt{1-x^2} dx = 4 \int_0^1 \sqrt{1-x^2} dx$$

antiderivative?

"trig substitution" is substitution in reverse.

$$\text{let } x = \sin \theta \rightarrow \begin{matrix} x=0 : \theta=0 \\ x=1 : \theta=\pi/2 \end{matrix}$$

$$dx = \cos \theta d\theta$$

transform integral:

$$A = 4 \int_0^{\pi/2} \underbrace{\sqrt{1-\sin^2 \theta}}_{+\cos \theta} \cos \theta d\theta = 4 \int_0^{\pi/2} \frac{\cos^2 \theta}{2} d\theta \quad \left(\begin{matrix} \text{half/double} \\ \text{angle identity} \end{matrix} \right)$$

$$= 2 \int_0^{\pi/2} 1 - \cos 2\theta d\theta$$

$$= 2 \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_0^{\pi/2} = 2 \left(\frac{\pi}{2} - \frac{1}{2} \sin \pi \right) - 0 = \pi = \pi(1)^2 \checkmark$$

trig substitutions like this lead to integrals of products of powers of sines and cosines, hence "trig integrals" in section 7.2.

7.1 Integration by parts etc

(2)

What about section 7.4 partial fractions?
(No so much fun like the video states!)

Example $\int \frac{x^2 - x + 6}{x^3 + 3x} dx$

quotient of polynomials
(rational function)

convert to partial fractions
 $= \int \frac{-x}{x^2+3} - \frac{1}{x^2+3} + \frac{2}{x} dx$

whole "fraction" is a sum of "partial fractions" accomplished by matrix algebra calculation.

each term easily integrated.

$\sim \int \frac{du}{u} \sim \ln u$ \arctan $2 \ln x$

$\int \frac{dx}{x^2+3} \xrightarrow{\substack{\text{let } x = \sqrt{3}u \\ dx = \sqrt{3}du}} \int \frac{\sqrt{3}du}{3u^2+3} = \frac{1}{\sqrt{3}} \int \frac{du}{1+u^2} = \frac{1}{\sqrt{3}} \arctan u + c = \frac{1}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right)$

$= -\frac{1}{2} \ln(x^2+3) - \frac{1}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + 2 \ln x + C$

Maple produces the antiderivatives when we need them.
The point is to use those in answering mathematical questions.

7.1

Integration by Parts etc

3

example

$$\int_0^\pi e^{\cos t} \sin 2t \, dt$$

$u = \cos t$
 $du = -\sin t \, dt$

$2 \underbrace{\cos t}_{u} \sin t$

clearly needs u-sub first

$$t=0 \rightarrow u = \cos 0 = 1$$

$$t=\pi \rightarrow u = \cos \pi = -1$$

$$= \int_{t=0}^{t=\pi} e^u 2u \, (du)$$

first example we did!

$$= -2 \int_1^{-1} u e^u \, du = 2 \int_{-1}^1 u e^u \, du = 2(u-1)e^u \Big|_{-1}^1$$

$$= 2(1-1)e^1 - 2(-1-1)e^{-1} = 4e^{-1} \approx \dots$$

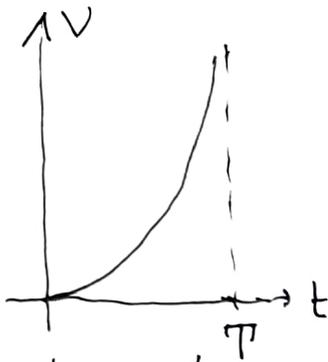
(we saw this in the HW due today.)

Question. When do we ever need integrals of these weird functions, like $\ln x$ for example?

Stewart: 7.1.68

7.1 exercise 68: modeling rocket-liftoff with variable thrust

4



model good as long as not too close to $t = \tau$

The velocity has 2 terms:

$$v(t) = \underbrace{-gt}_{\text{gravity pulls down}} - \underbrace{v_e \ln\left(1 - \frac{r}{m}t\right)}_{\text{rocket thrust up}}$$

< 1 so $\ln(\cdot) < 0$
(2 forces compete.)

$g = 9.8 \text{ m/sec}^2$

$m = \text{mass of rocket}$

$r = \text{rate of mass loss of fuel}$

$v_e = \text{velocity of exhaust gas}$

} variable parameters set by web assign.

dimensions:

$1 - \frac{r}{m}t = 1 - \frac{t}{\tau}$ requires $\tau = \frac{m}{r} \sim \frac{\text{mass}}{\text{mass/time}} = \text{time characterizing thrust}$
dimensionless

How long for "falling velocity" to equal v_e ? $gt = v_e \rightarrow t = \frac{v_e}{g} \equiv \tau$
This characterizes the gravity time scale

$$\frac{v}{v_e} = -\frac{t}{\tau} - \ln\left(1 - \frac{t}{\tau}\right)$$

2 time scales, only ratio matters
 $w = \frac{t}{\tau} \rightarrow t = w\tau$ [w = dimensionless time from τ]

$= -\left(\frac{\tau}{\tau}\right)w - \ln(1-w)$ ← only one parameter family of shaped curves
 $\frac{\tau}{\tau} < 1$ gravity timescale must be longer than thrust timescale to lift off
(see Maple)

Goal: Integrate velocity to get height function of time

Need $\int \ln x dx = x(\ln x - 1) + c$

Result for height:

$$y = -\frac{1}{2} \underbrace{(v_e \tau)}_{\text{length}} \left(\frac{t}{\tau}\right)^2 + \underbrace{(v_e \tau)}_{\text{length}} \left(1 - \frac{t}{\tau}\right) \ln\left(1 - \frac{t}{\tau}\right)$$

7.1 Integration by Parts etc

5

How do we get this? First a u-sub:

$$y = \int -gt - ve^{\tau} \left(1 - \frac{t}{\tau}\right) dt$$

$$= -\frac{1}{2}gt^2 - ve^{\tau} \int \ln\left(1 - \frac{t}{\tau}\right) dt$$

$$u = 1 - \frac{t}{\tau} \quad du = -\frac{dt}{\tau} \rightarrow dt = -\tau du$$

$$= -\frac{1}{2}gt^2 + ve^{\tau} \int \ln u \, du$$

$$u (\ln u - 1) = \left(1 - \frac{t}{\tau}\right) \left[\ln\left(1 - \frac{t}{\tau}\right) - 1\right]$$

$$= -\frac{1}{2}gt^2 + ve^{\tau} \left(1 - \frac{t}{\tau}\right) \left[\ln\left(1 - \frac{t}{\tau}\right) - 1\right]$$

$$gt^2 = \frac{ve^{\tau}}{\tau} t^2 = ve^{\tau} \left(\frac{t}{\tau}\right)^2$$

$$= -\frac{1}{2} (ve^{\tau}) \left(\frac{t}{\tau}\right)^2 + (ve^{\tau}) \left(1 - \frac{t}{\tau}\right) \left[\ln\left(1 - \frac{t}{\tau}\right) - 1\right]$$

↑
gravity
timescale

thrust timescale

clearly Maple appropriate for such long formulas to evaluate for specific values of parameters.
at $t = 60$ (1 minute after take off.)