

7.1 Integration by Parts

①

CHAPTER 7 WARNING

7.1 Integration by Parts
7.2 Trig integrals
7.3 Trig substitution
7.4 Partial fractions

} superseded by modern technology!

In the age of computer algebra systems for calc students, the only useful "technique of integration" worth teaching is "u-substitution" since transforming definite integrals is important and it gives us a peek into the rabbit hole of integration techniques. There is no general approach to antidifferentiation and in our class we will only use the simple rules & u-substitution and Maple for all other antiderivatives.

Integration by parts gives another window into this bottomless pit of techniques but is also important for deriving differential equations in advanced math, physics and engineering, so is worth being exposed to, but once done, we will not use it later in the class.

Rules of differentiation re-expressed "backwards" give us rules of integration. The chain rule gave us "u-substitution".
The product rule gives us "integration by parts".

7.1

Integration by Parts

(2)

The product rule:

function notation: $\frac{d}{dx}(f(x)g(x)) = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$

dependent variable notation: $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$

solve for final term:

$$f(x)\frac{dg(x)}{dx} = \frac{d}{dx}(f(x)g(x)) - \frac{df(x)}{dx}g(x)$$

integrate:

$$\int \underbrace{f(x)}_u \underbrace{\frac{dg(x)}{dx}}_{dv} dx = \int \frac{d}{dx}(f(x)g(x)) dx - \int \underbrace{g(x)}_v \underbrace{\frac{df(x)}{dx}}_{du} dx$$
$$\underbrace{f(x)}_u \underbrace{g(x)}_v = \int u dv = uv - \int v du$$

so what?

When an integrand is a product (for which u-sub is not the obvious choice), we can try to identify the factors as above to see if the integral on the RHS may be simpler to integrate.

examples show how to apply this "technique".

7.1] Integration by Parts

(3)

example $\int x e^x dx$

product integrand — we can integrate and differentiate both factors so ... which do we pick for u ?

try both to see what happens.

$$\int \underbrace{x}_u \underbrace{e^x dx}_{dv} = \underbrace{x}_u \underbrace{e^x}_v - \int \underbrace{e^x}_v \underbrace{dx}_{du} = x e^x - e^x + C = (x-1) e^x + C$$

$u=x \rightarrow dv=e^x dx$
 $du=dx \rightarrow v=\int e^x dx = e^x$
 yes!

$$\int \underbrace{e^x}_u \underbrace{x dx}_{dv} = \underbrace{\frac{x^2}{2}}_u \underbrace{e^x}_v - \int \underbrace{\frac{x^2}{2}}_v \underbrace{e^x dx}_{du}$$

$u=e^x \rightarrow dv=x dx$
 $du=e^x dx \rightarrow v=\frac{x^2}{2}$
 worse! more complicated.
 NO GO! STOP.

remark 1 we don't need constants of integration for v ! we want to pick u & v to lead to a simpler alternative integral.

In this case differentiation lowers the power of x by one, so if we have higher powers, we can

iterate the process.

Note: $\int_0^1 x e^x dx = (x e^x - e^x) \Big|_0^1 = 1e^1 - e^1 - (0e^0 - e^0) = 1$

$\int_0^1 x e^x dx = x e^x \Big|_0^1 - \int_0^1 e^x dx$ applies directly to first step.

7.1 Integration by Parts

4

example $\int \underbrace{x^2}_u \underbrace{e^x}_{dv} dx = \underbrace{x^2}_u \underbrace{e^x}_v - \int \underbrace{e^x}_v \underbrace{(2x dx)}_{du}$

$u = x^2 \rightarrow du = 2x dx$
 $dv = e^x dx \rightarrow v = \int e^x dx = e^x$

$2 \int x e^x dx$
 $(x-1)e^x$ done!

$= x^2 e^x - 2(x-1)e^x + C$
 $= (x^2 - 2x + 2) e^x + C \quad \checkmark$

example $\int \underbrace{x^3}_u \underbrace{e^x}_{dv} dx = \underbrace{x^3}_u \underbrace{e^x}_v - \int \underbrace{e^x}_v \underbrace{(3x^2 dx)}_{du}$

$u = x^3 \rightarrow du = 3x^2 dx$
 $dv = e^x dx \rightarrow v = e^x$

$3 \int x^2 e^x dx$
 $3(x^2 - 2x + 2)e^x + C$

$= (x^3 - 3x^2 + 6x - 6) e^x + C$



example $\int \underbrace{x^n}_u \underbrace{e^x}_{dv} dx = \underbrace{x^n}_u \underbrace{e^x}_v - \int \underbrace{e^x}_v \underbrace{(n x^{n-1} dx)}_{du}$

$u = x^n \rightarrow du = n x^{n-1} dx$
 $dv = e^x dx \rightarrow v = e^x$

$= x^n e^x - n \int x^{n-1} e^x dx$

"reduction formula"

not for us!

programmed into Maple!

7.1 Integration by Parts

(5)

All inverse functions have derivative rules which can be reversed by integration by parts:
 $\ln = \exp^{-1}$, arctrig functions.

example $\int \underbrace{\ln x}_u \underbrace{dx}_{dv} = \underbrace{x \ln x}_u \underbrace{- \int x \left(\frac{1}{x} dx\right)}_{dv}$

$u = \ln x \rightarrow dv = dx$
 $du = \frac{1}{x} dx \leftarrow v = x$

$\int 1 dx = x$

$= x \ln x - x + C$
 $= x(\ln x - 1) + C$

example $\int \underbrace{\arctan x}_u \underbrace{dx}_{dv} = x \arctan x - \int x \left(\frac{1}{1+x^2}\right) dx$

$u = \arctan x \rightarrow dv = dx$
 $du = \frac{1}{1+x^2} dx \leftarrow v = x$

$\frac{1}{2} \int \frac{2x dx}{1+x^2}$

$\int \frac{1}{w} dw = \ln|w| = \ln(1+x^2)$

$= x \arctan x - \frac{1}{2} \ln(1+x^2) + C$

$\int_0^1 \arctan x dx = (x \arctan x - \frac{1}{2} \ln(1+x^2)) \Big|_0^1$

$= \underbrace{1 \arctan 1}_{\pi/4} - \frac{1}{2} \ln \left(\frac{1+1}{2}\right) - \underbrace{0 \arctan 0}_0 + \frac{1}{2} \underbrace{\ln(1+0)}_0$

$= \pi/4 - \frac{1}{2} \ln 2 \approx 0.4388 > 0 \checkmark$

> 0

at least check sign

7.1

Integration by Parts

6

U-sub type integrals are also product integrand cases.

$$\int x \sqrt{1-x^2} dx \quad \text{obvious "u-sub": } u=1-x^2$$

$$= \int \underbrace{\sqrt{1-x^2}}_u \underbrace{x dx}_{dv} = \frac{x^2}{2} \sqrt{1-x^2} - \int \frac{x^2}{2} \left(\frac{-x dx}{\sqrt{1-x^2}} \right) + \frac{1}{2} \int \frac{x^3}{\sqrt{1-x^2}} dx$$

$$u = (1-x^2)^{1/2}$$

$$dv = x dx$$

$$du = \frac{1}{2} (1-x^2)^{-1/2} (-2x) dx \leftarrow v = \frac{x^2}{2}$$

$$= \frac{-x}{\sqrt{1-x^2}} dx$$

much worse!

The idea is to try to simplify the integral needed to be evaluated through this process.

$$\int x^3 \sqrt{1-x^2} dx = \int \underbrace{x^2}_u \underbrace{(1-x^2)^{1/2} x dx}_{dv}$$

$$u = x^2 \quad dv = -\frac{1}{2} \frac{(1-x^2)^{1/2} (-2x dx)}{w^{1/2} dw}$$

$$du = 2x dx \quad v = -\frac{1}{2} \frac{w^{3/2}}{3/2} = -\frac{1}{3} (1-x^2)^{3/2}$$

$$= -\frac{x^2}{3} (1-x^2)^{3/2} + \frac{2}{3} \int x (1-x^2)^{3/2} dx$$

doable by u-sub

BUT so is original!

IOP not useful here.