

## 6.5 Average value of a function ①

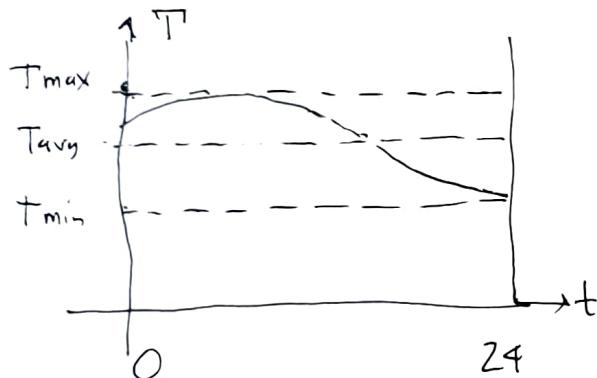
We all know how to average a set of numbers  $\{y_1, \dots, y_n\}$ .  
We add them up and divide by their number:

$$y_{\text{avg}} = \frac{\sum_{i=1}^n y_i}{n} \quad \left. \begin{array}{l} \text{sum of values} \\ \text{number of values} \end{array} \right\}$$

It is clear from the process that  $y_{\text{avg}}$  is some "intermediate value":  $y_{\min} \leq y_{\text{avg}} \leq y_{\max}$

multiplying through:  $\sum_{i=1}^n y_i = n(y_{\text{avg}})$  same sum as if we replaced all the numbers by  $y_{\text{avg}}$

But suppose we have a continuous function over an interval like the temperature over a 24 hour period



The "average" temperature is a familiar idea & it somehow averages out the temperature curve to some intermediate value.

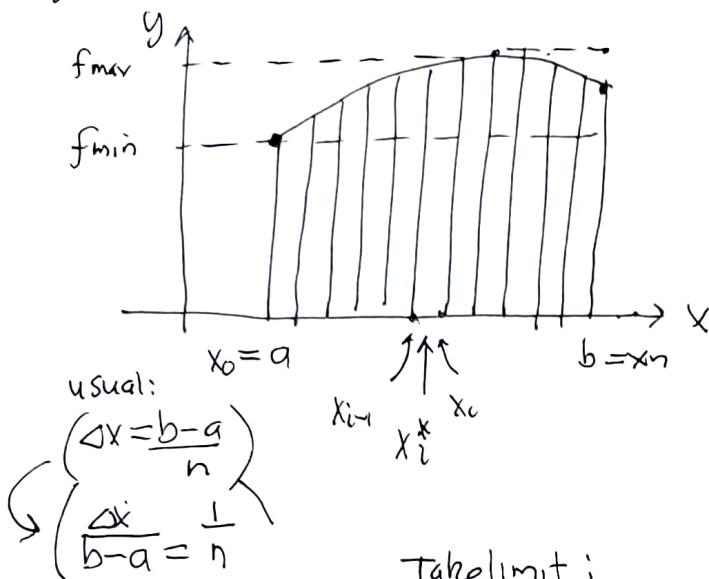
How?

We could sample the temperature once an hour ( $n=24$ ) or once per minute ( $n=24 \times 60$ ) and just average all the sampled values. Obviously as we shorten the time intervals for sampling we get a better result. This is just the Riemann sum approximation in action!

## 6.5 Average value of a function

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general situation:



$f$  continuous on  $a \leq x \leq b$

sample function on each subinterval

$f(x_i^*)$  stay midpoint value  
to avoid bias

average the  $n$  values:

$$f_{\text{avg}} \approx \frac{\sum_{i=1}^n f(x_i^*) \Delta x}{n} = \frac{\sum_{i=1}^n f(x_i^*) \Delta x}{b-a}$$

Take limit:

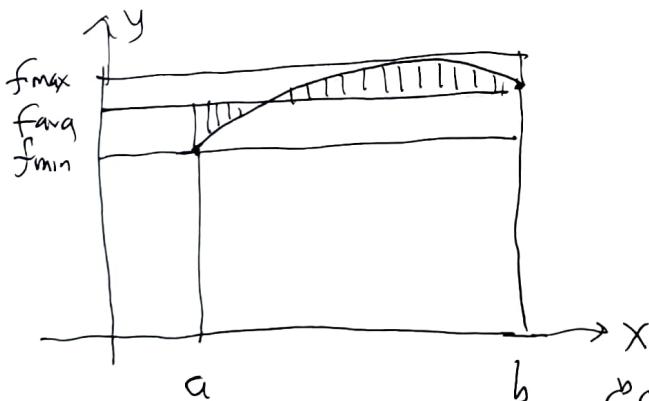
$$f_{\text{avg}} = \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n f(x_i^*) \Delta x}{b-a} = \frac{\int_a^b f(x) dx}{b-a}$$

"sum" of the sampled values  
measure of how many  $x$  values

and note by multiplying through

$$\int_a^b f(x) dx = f_{\text{avg}}(b-a) = \text{area of rectangle with top } y = f_{\text{avg}}$$

(in general "signed" area!)



area below balances area above  $y = f_{\text{avg}}$ .

clear that  $f_{\min} \leq f_{\text{avg}} \leq f_{\max}$

why?

$$\frac{\int_a^b f_{\min} dx}{b-a} \leq \frac{\int_a^b f(x) dx}{b-a} \leq \frac{\int_a^b f_{\max} dx}{b-a}$$

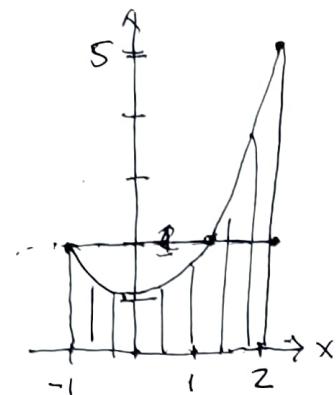
$$f_{\min} \left( \frac{b-a}{b-a} \right) \quad f_{\text{avg}} \quad f_{\max} \left( \frac{b-a}{b-a} \right)$$

6.5

Average value of a function

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math example  $f(x) = 1 + x^2$  for  $-1 \leq x \leq 2$



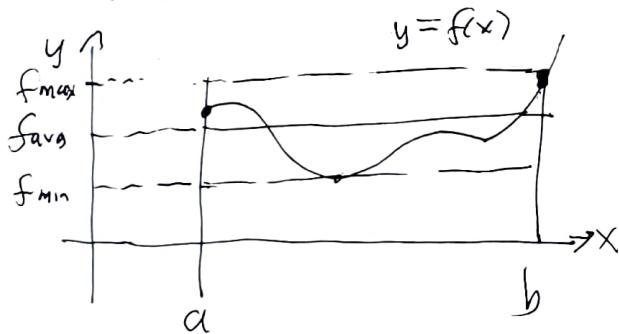
$$\begin{aligned}
 f_{\text{avg}} &= \frac{1}{3} \int_{-1}^2 1 + x^2 \, dx = \frac{1}{3} \left( x + \frac{x^3}{3} \right) \Big|_{-1}^2 \\
 &= \frac{1}{3} \left[ \left( 2 + \frac{8}{3} \right) - \left( -1 - \frac{1}{3} \right) \right] \\
 &= \frac{1}{3} \left[ 2 + \underbrace{\frac{8}{3} + 1 + \frac{1}{3}}_3 \right] = \frac{6}{3} = 2 \quad \text{intermediate value}
 \end{aligned}$$

$$1 \leq f(x) \leq 5$$

But also obvious that  $f$  crosses the avg value line.

$$\begin{aligned}
 f(x) = f_{\text{avg}} : \quad 1 + x^2 &= 2 \\
 x^2 &= 1 \\
 x = \pm 1 &\rightarrow y = 1 + (\pm 1)^2 = 2 \quad \checkmark
 \end{aligned}$$

This is called "the mean value theorem".



any continuous function on an interval  
must assume every intermediate  
value between min & max  
so must assume avg value  
at least once, may be more.

But solving  $f(x) = f_{\text{avg}}$  may require numerical solution

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Average velocity

Even before calculus we learn about average velocity over a time interval

$$\text{avg vel} = \frac{\text{distance traveled}}{\text{time elapsed}} \quad \text{over interval } t_1 \leq t \leq t_2$$

$$v_{\text{avg}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{\int_{t_1}^{t_2} s'(t) dt}{t_2 - t_1} \quad \text{net change theorem}$$

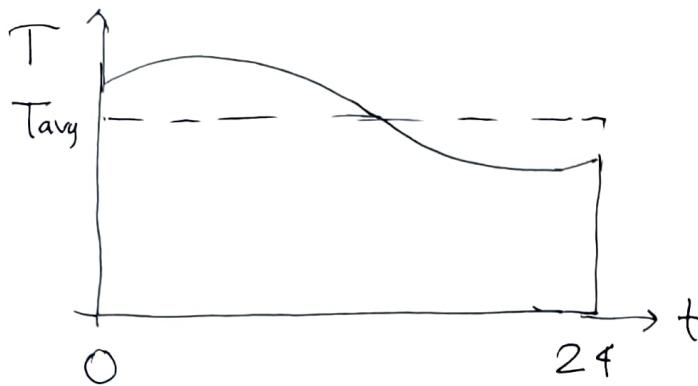
$$= \frac{\int_{t_1}^{t_2} v(t) dt}{t_2 - t_1} = v_{\text{avg}}! \quad \text{agrees with our integral definition.}$$

Intuition: at some point in time we actually have to be moving at the average speed!

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Average value of a function

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Average temperature

24 hour period say  
from 5am to 5am

see Maple worksheet

$$T(t) = 72 \left( e^{-\frac{(t-2)}{4(24)}} + 0.1 \sin \frac{2\pi(t-2)}{24} \right)$$

$$T_{\text{avg}} = \frac{1}{24} \int_0^{24} T(t) dt = \frac{1}{24} \int_0^{24} 72 \left( e^{-\frac{(t-2)}{4(24)}} + 0.1 \sin \frac{2\pi(t-2)}{24} \right) dt$$

↓                  ↓  
easy              easy

but really? do we  
need to do this?

$$\begin{aligned} & \left[ \int e^{-a(t-b)} dt \right] \\ &= -\frac{1}{a} e^{-a(t-b)} + C \end{aligned}$$

$$\approx 65.046 \approx 65^\circ$$

Maple

When? The graph shows  $t \approx 13$ .

so  $5+13 \rightarrow 6\text{pm}$

see Maple worksheet

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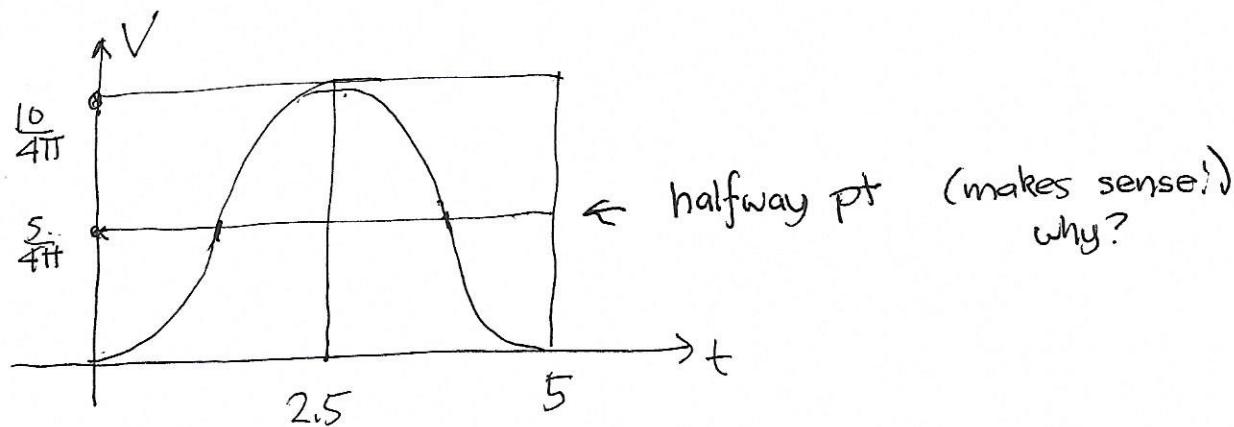
6.5.23 from 5.5.83

Models amount of inhaled air during one 5 second breath cycle of inhaling then exhaling.

result:  $V = \int_0^t \frac{1}{2} \sin \frac{2\pi u}{5} du$

$$= \left( \frac{5}{2\pi} \right) \left( \frac{1}{2} \right) \left( -\cos \frac{2\pi u}{5} \right) \Big|_0^t$$

$$= \frac{5}{4\pi} \left( 1 - \cos \frac{2\pi t}{5} \right) \text{ liters } (\approx \text{quarts!})$$



$$V_{\text{avg}} = \frac{1}{5} \int_0^5 \frac{5}{4\pi} \left( 1 - \cos \frac{2\pi t}{5} \right) dt$$

$$= \frac{1}{5} \left( \frac{5}{4\pi} \right) \left( t - \frac{5}{2\pi} \sin \frac{2\pi t}{5} \right) \Big|_0^5$$

$$= \frac{1}{5} \left( \frac{5}{4\pi} \right) \left( 5 - \frac{5}{2\pi} \sin 2\pi + \frac{5}{2\pi} \sin 0 \right)$$

$$= \frac{5}{4\pi} \text{ halfway point (obvious from symmetry)}$$

$\approx 0.40$  : useless unless numerical value for interpretation!  
(about half a liter)