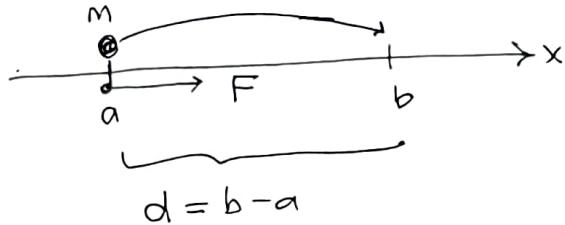


6.4 Work in 1-d motion

Work done by a force is a change of energy, and energy powers our civilization, so it is not a bad idea to look at this concept.

①

pre-calc work



Consider 1-d motion along an axis. Let a force F move a mass M over a displacement d .

The work done by the force to move the mass over this displacement:

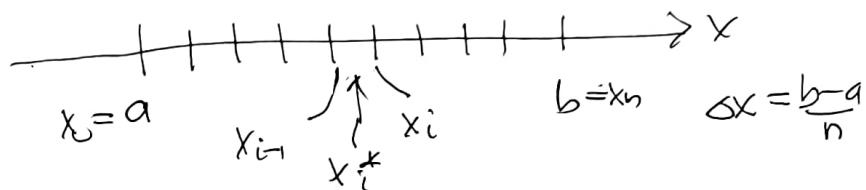
$$W = Fd \quad \text{units: force} \times \text{length}$$

calc work variable force $F(x)$

The product is replaced by the integral

$$W = \int_a^b F(x) dx$$

Why? Riemann.



$$\Delta W_i \approx F(x_i^*) \Delta x \quad \text{product for small interval}$$

$$W \approx \sum_{i=1}^n \Delta W_i$$

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n F(x_i^*) \Delta x = \int_a^b F(x) dx \quad \text{easy!}$$

We "sum up" the individual products over small intervals \Rightarrow integration

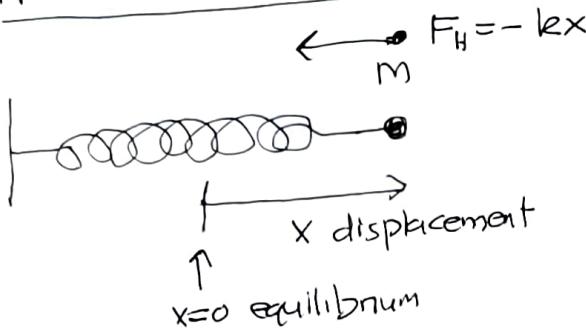
2 kinds of common examples

- 1) a nonconstant force, fixed displacement (DONE! above)
- 2) a constant force, variable displacement (more interesting, needs Riemann)

G4

Work in 1-d motion

(2)

Nonconstant Force examplesHooke's Law mass spring system

(prototype for many oscillating systems)

restoring force:

stretch $x > 0$ — pulls back \rightarrow to equilibriumcompress $x < 0$ — pushes backWork done against the spring to move the mass from a to $b > a$

$$W = \int_a^b F dx = \int_a^b kx dx = \frac{1}{2} kx^2 \Big|_a^b$$

+ opposing spring

$$= \frac{1}{2} kb^2 - \frac{1}{2} ka^2$$

$$= \Delta U \quad \text{where}$$

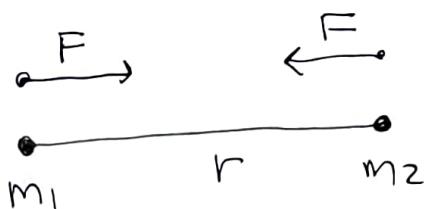
$$U = \frac{1}{2} kx^2 \quad \text{"potential energy" stored in spring}$$

If you "release" a (stretched) spring
(compressed)It will (pull)
(push) back the mass, doing work in moving it.

6.4

Work in 1-d motion

(3)

Newton's law of gravity

$$F = \frac{Gm_1 m_2}{r^2}$$

inverse square
force
of attraction

Work done by gravitational force to bring 2 masses closer together from $r = b > a$ to $r = a$

$$W = \int_a^b \frac{Gm_1 m_2}{r^2} dr = -\frac{Gm_1 m_2}{r} \Big|_a^b = G\frac{m_1 m_2}{a} - G\frac{m_1 m_2}{b} > 0$$

$\int r^{-2} dr = -r^{-1} + C$

$= \Delta U \quad \text{where} \quad U = G\frac{m_1 m_2}{r}$

potential energy
stored in the system
ready to be released

If takes energy to pull apart 2 masses.

ΔU is how much energy it takes to increase the separation opposing the attraction.

6.4

Work in 1-d motion

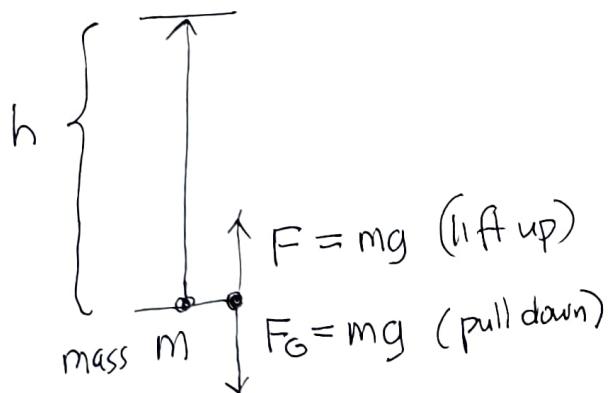
(4)

constant force, variable displacements: gravity and lifting mass

precalc example

Earth surface gravity

lift mass up:



Work down opposing downward
constant g -force

$$W = F h = \underbrace{mg}_{\text{weight}} h = wh$$

g = gravitational
acceleration.

units?

$$F = m a = m \frac{d^2 x}{dt^2}$$

$\underbrace{m}_{\text{mass}} \underbrace{\frac{d^2 x}{dt^2}}_{\text{acceleration}}$

$\frac{\text{mass} * \text{length}}{\text{time}^2}$

US no named mass units, we use weight instead
work units: weight * length \sim ft-lbs !!

Metric MKS (meters-kilograms-seconds)

work units weight * length $\underbrace{\text{newton-meters}}_{= (\frac{\text{kg} \cdot \text{meter}}{\text{sec}^2})} = \text{joules}$
= force unit

Who cares? In US units we use weight instead of mass

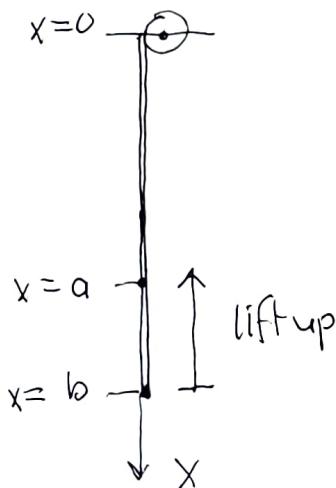
6.4

Work in 1-d motion

(5)

Variable displacement : surface gravity

Lift extended object in vertical direction (against gravity)

example : Raise, lift, or "wind up" a rope or cable to a certain level.

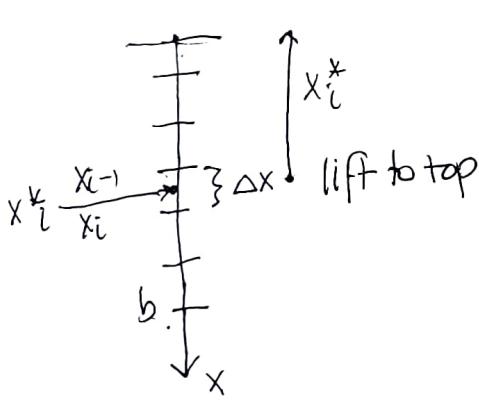
cable "hanging down"
so suggests letting
 x measure down from top
(not necessary)

Key point: different pieces of the cable are lifted different heights even though force is constant.

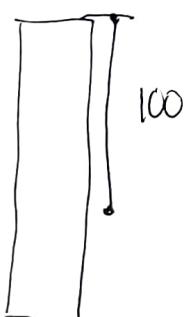
Need Riemann to sum them up.

work = weight * distance
so we need a weight density for the cable:

$$\cdot (\Delta m) g = \rho \underbrace{\Delta x}_{\substack{\text{increment} \\ \text{of weight}}} \quad \begin{array}{l} \text{weight / length} \\ (\text{constant}) \end{array}$$



explicit numbers



$$\Delta W_i = (\rho \Delta x) x_i^* \quad \text{weight * distance}$$

$$W = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho x_i^* \Delta x = \int_0^b \rho x \, dx$$

A 200 lb cable is 100 ft long and hanging from the top of a building. How much work is required to lift the cable to the top?

$$\rho = \frac{200 \text{ lb}}{100 \text{ ft}} = 2 \text{ lb/ft}$$

$$W = \int_0^{100} \rho x \, dx = \rho \frac{x^2}{2} \Big|_0^{100} = 10^4 = 10,000 \text{ ft-lbs}$$

Variations: 1) only lift partway \rightarrow limits of integration
2) add mass to end of cable

6.4

Work in 1-d motion

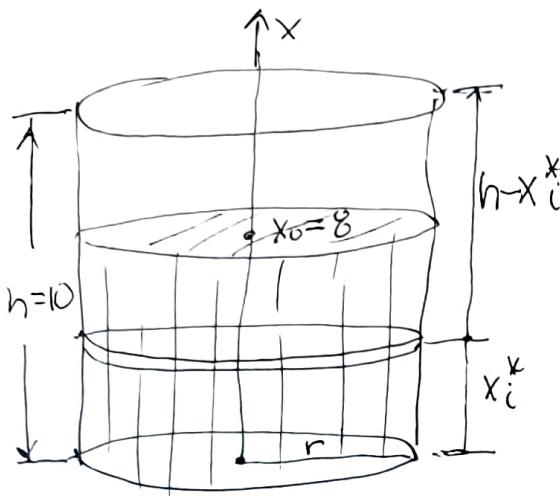
(6)

2-d lifting [NOT FOR HW, QUIZ, TEST]

These are excellent examples of Riemann integral formula building.

[cylindrical tank, pump liquid] to top of tank (and then out)

$$\text{weight density: } \rho = 20 \text{ lb/ft}^3$$



} each layer is lifted to top (pump, whatever)

weight increment:

$$\Delta w_i = \rho \Delta V_i = \rho A \Delta x$$

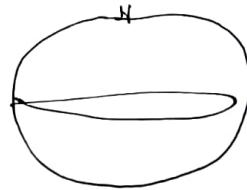
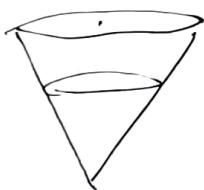
$$\Delta V_i = (\pi r^2) \Delta x = A \Delta x$$

$$\Delta W_i = \Delta w_i (h - x_i^*) = \rho A (h - x_i^*) \Delta x$$

$$\bar{W} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho A (h - x_i^*) \Delta x = \int_0^{x_0} \rho A (h - x) dx$$

= ... plug in numbers

Variations:
- variable horizontal plane cross-sectional area $A(x)$
- partially lower liquid level (limits of integration)



odd shapes

but this is not a class in physics or
chemical engineering so ...

nevermind

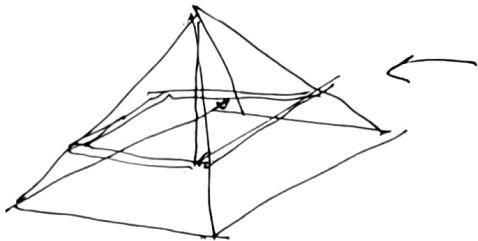
6.4

Work in 1-d motion

7

Fun

Build a pyramid!

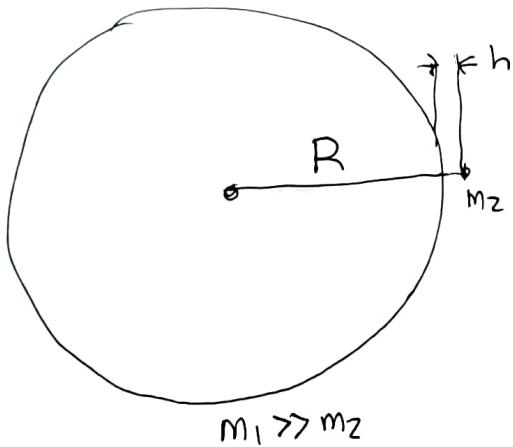


each layer of rock blocks must be lifted from ground level to final position

same as pumping problem but solid instead of liquid.

problem 34. Great Pyramid.

surface gravity?



gravity acts as though all mass is at the center

work needed to increase separation from $r=R$ to $r=R+h$?

$$\Delta U = \frac{Gm_1m_2}{R} - \frac{Gm_1m_2}{R+h} = Gm_1 \left(\frac{1}{R} - \frac{1}{R+h} \right) m_2$$

$$= Gm_1 \left(\frac{R+h-R}{R(R+h)} \right)^{\circ} m_2 = \frac{Gm_1}{R^2} \frac{h}{1+h/R} m_2$$

$$= \left(\frac{Gm_1}{R^2} \right) \frac{m_2 h}{1+h/R} \approx (m_2 g) h !$$

≈ 0 if $h \ll R$

Newton's falling apple also explains planetary motion!