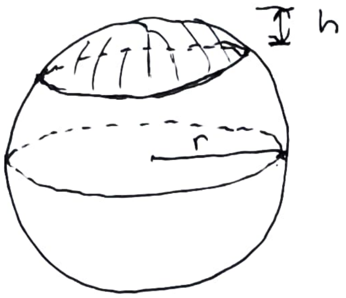
G2 translating word descriptions of solids to function graphs ①

When a solid is described by its geometry, we need to set up a diagram in the x-y plane to apply our volume approach.

Example

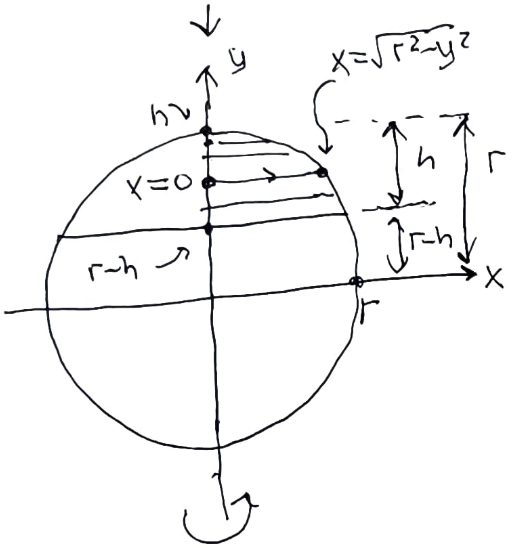
Find the volume of a cap of a sphere with radius r and height h.



Solution: This is a volume of revolution about the vertical axis as drawn and can be obtained by rotating part of a circle as shown in the diagram. y is the independent variable.

$$x^2+y^2=r^2 \to x=\pm\sqrt{r^2-y^2}$$
$$\to \sqrt{r^2-y^2}$$

only right half of circle needed to rotate.



The plane cross-sectional area is

$$A(y)=\pi\left(\sqrt{r^2-y^2}\right)^2=\pi(r^2-y^2)$$

and $y=r-h\,..\,r$.

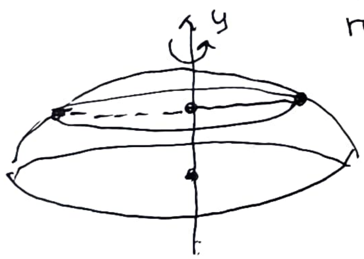
$$\text{so } V=\int_{r-h}^{r}\pi(r^2-y^2)\,dy$$

$$=\pi\left(r^2y-y^3/3\right)\Big|_{r-h}^{r}$$

$$=\pi\left(r^3-r^3/3-\left(r^2(r-h)-(r-h)^3/3\right)\right)$$

$$=\pi\left(\cancel{r^3}-\tfrac{1}{3}r^3-\cancel{r^3}+r^2h+\tfrac{(r-h)^3}{3}\right)$$

linear cross-section rotates around to sweep out a circular disk



$$=\frac{\pi}{3}\left((r-h)^3+3r^2h-r^3\right)$$

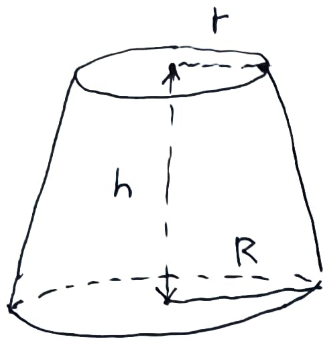
$$=\frac{\pi}{3}\left(\cancel{r^3}-\cancel{3r^2h}+3rh^2-h^3+\cancel{3r^2h}-\cancel{r^3}\right)$$

$$=\frac{\pi}{3}(3rh^2-h^3)=\frac{\pi h^2}{3}(3r-h)=\boxed{\pi h^2\left(r-\frac{h}{3}\right)}$$

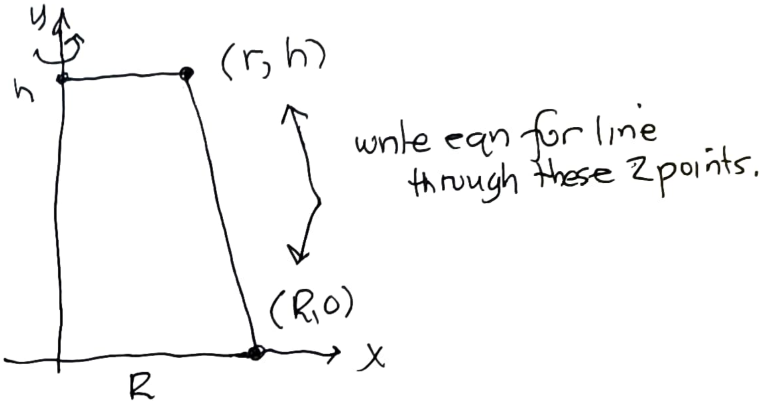
Note $V=0$ when $h=0$, while $V=\pi r^2\left(r-\frac{r}{3}\right)=\frac{2\pi r^3}{3}$ when $h=r$, so both limiting cases are correct.

6.2 translating volumes into x-y plane diagrams

example 6.2.48 Find the volume of a "frustrum" of a right circular cone with height h, lower base radius R and top radius r.



bob hint:



Now you try this.

Set up integral but let Maple evaluate it AND simplify it.

Does your result have the two limiting cases $r=0 \rightarrow V =$ cone formula and $h=0 \rightarrow V=0$?

[What about the case $r=R$ making this a cylinder?]