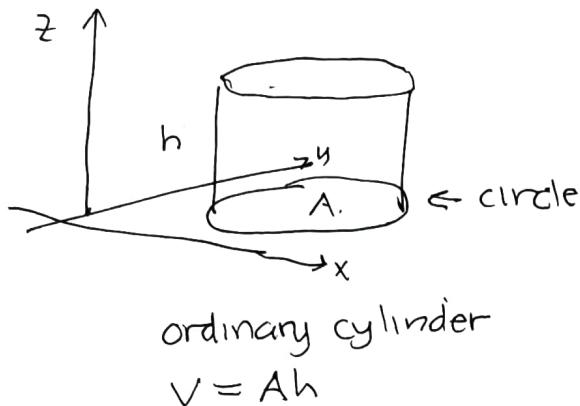


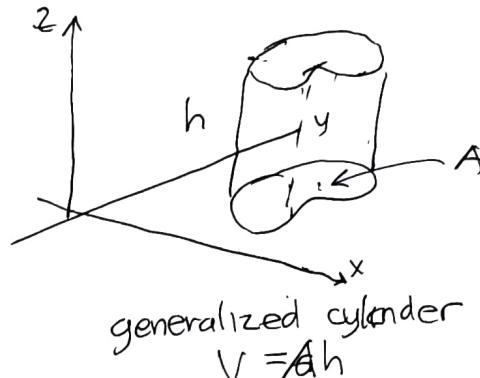
## 6.2 volumes (of revolution etc)

①

If we take any region of the plane with area  $A$  and sweep it in space perpendicular to the region, it sweeps out a "generalized cylinder!"



ordinary cylinder  
 $V = Ah$

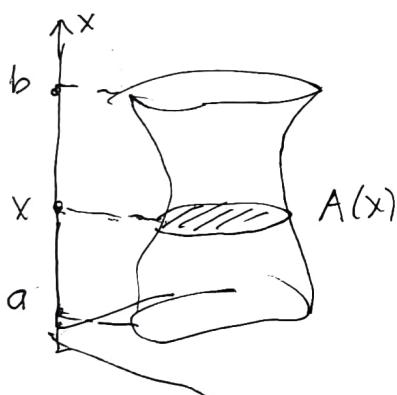


generalized cylinder  
 $V = Ah$

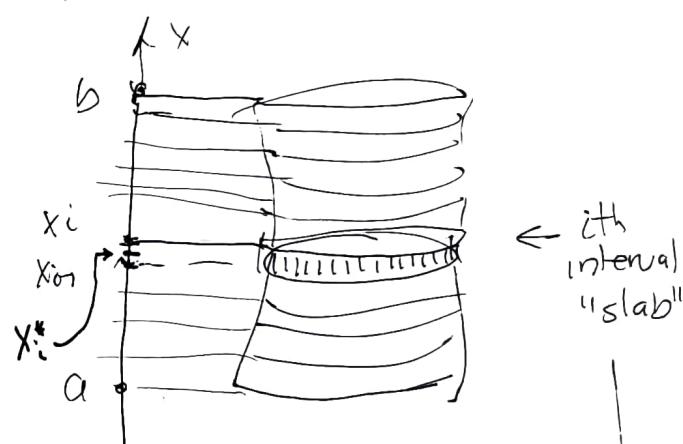
All horizontal plane cross-sections are identical.

If we allow the plane cross-sections to be variable regions, we have to integrate the variable area instead of multiplying by the height.

Let  $x$  be the axis describing this one-parameter family of plane cross-sections, with variable shape.



→ Riemann approach



$$\Delta x = \frac{b-a}{n} \rightarrow \Delta A_i \approx A(x_i^*) \Delta x$$

$\underbrace{\Delta A_i}_{\text{base}} \approx \underbrace{A(x_i^*)}_{\text{height}}$   
= exact volume of slab with shape at  $x_i^*$  and thickness  $\Delta x$

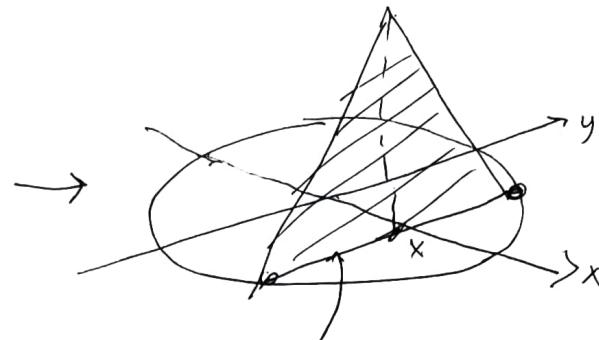
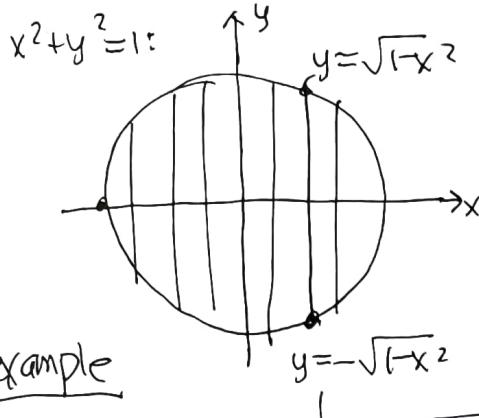
$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta A_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i^*) \Delta x$$

$$= \int_a^b A(x) dx$$

so as long as we can evaluate  $A(x)$  from the math description of the solid region, we just integrate it over the interval.

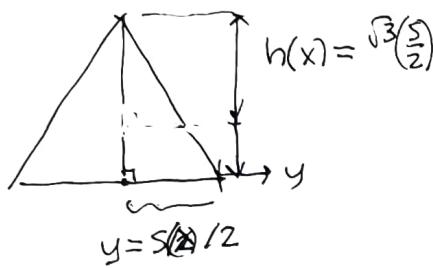
## 6.2) Volumes (of revolution etc)

(2)



example

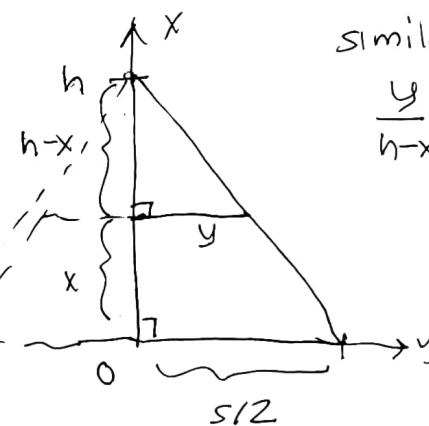
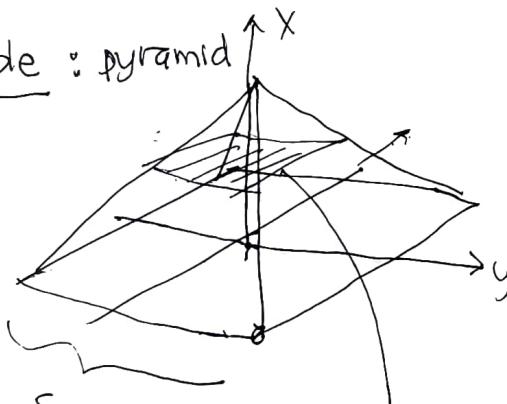
$$S(x) = 2\sqrt{1-x^2} \quad \text{side length equilateral triangle}$$



$$A(x) = \frac{1}{2} \underbrace{S(x)}_{\text{base}} \cdot \underbrace{\frac{\sqrt{3}}{2} S(x)}_{\text{height}} = \frac{\sqrt{3}}{4} S(x)^2 = \frac{\sqrt{3}}{4} \cdot 4(1-x^2) = \sqrt{3}(1-x^2)$$

$$\left. \begin{aligned} V &= \int_{-1}^1 \sqrt{3}(1-x^2) dx \\ &= \sqrt{3} \left( x - \frac{x^3}{3} \right) \Big|_{-1}^1 = (\text{symmetry}) \\ &= 2\sqrt{3} \left( x - \frac{x^3}{3} \right) \Big|_0^1 = 2\sqrt{3} \left( 1 - \frac{1}{3} \right) \\ &= \boxed{\frac{4}{3}\sqrt{3}} \end{aligned} \right\}$$

example : pyramid



similar triangles:

$$\frac{y}{h-x} = \frac{S/2}{h}$$

$$y = \frac{h-x}{h} \frac{S}{2}$$

$$A(x) = (2y)^2 = 4y^2 = 4 \left( \frac{S}{2} \frac{h-x}{h} \right)^2 = \frac{S^2(h-x)^2}{h^2}$$

$$\begin{aligned} V &= \int_0^h A(x) dx = \int_0^h \frac{S^2}{h^2} (h-x)^2 dx = \int_h^0 \frac{S^2}{h^2} u^2 (-du) \\ &\quad \left. \begin{aligned} u &= h-x & x=0 \rightarrow u=h \\ du &= -dx & x=h \rightarrow u=0 \end{aligned} \right\} \\ &= \int_0^h \frac{S^2}{h^2} u^2 du = \frac{S^2}{h^2} \frac{u^3}{3} \Big|_0^h = \frac{S^2}{h^2} \frac{h^3}{3} = \boxed{\frac{S^2 h}{3}} \end{aligned}$$

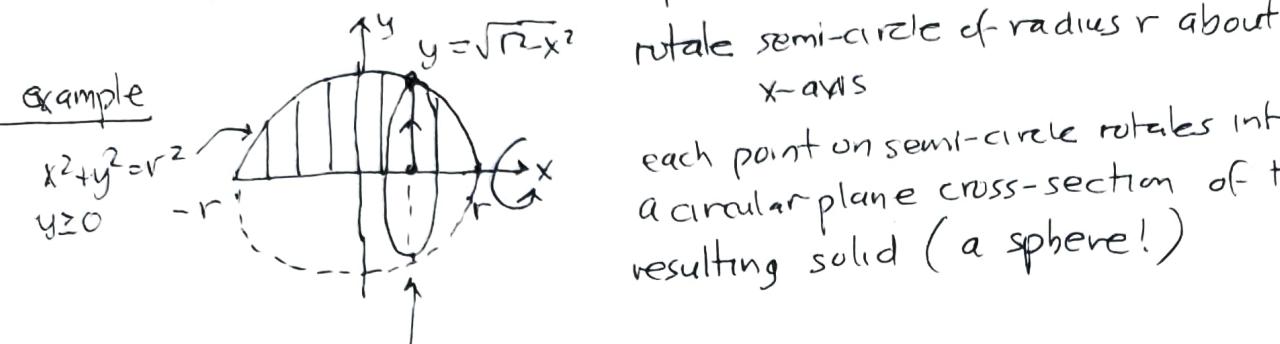
(=  $\frac{1}{3}$  volume of box of side  $S$ , height  $h$ )

## 6.2 Volumes (of revolution etc)

(3)

Volumes of revolution: special case in which plane cross-sections are circular disks with centers on an axis of symmetry

solids = result of rotating region of plane around an axis parallel to a coordinate axis in the plane



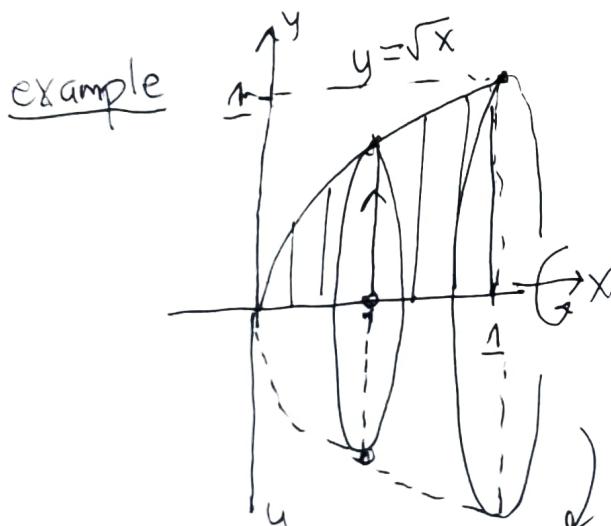
$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

$$V = \int_{-r}^r \pi(r^2 - x^2) dx = 2 \int_0^r \pi(r^2 - x^2) dx$$

$$= 2\pi \left( r^2x - \frac{x^3}{3} \right) \Big|_0^r = 2\pi \left( r^3 - \frac{r^3}{3} \right) = \frac{4\pi r^3}{3}$$

$$= \boxed{\frac{4\pi r^3}{3}}$$

(yay!)

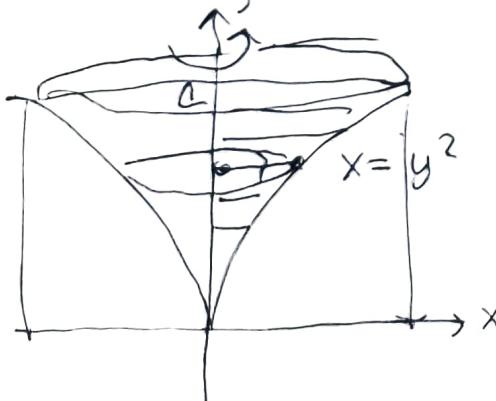


$$A(x) = \pi y^2 = \pi(\sqrt{x})^2 = \pi x$$

$$V = \int_0^1 \pi x dx = \pi \frac{x^2}{2} \Big|_0^1 = \frac{\pi}{2}$$

OR

same curve but about y-axis



$$A(y) = \pi x^2 = \pi(y^2)^2 = \pi y^4$$

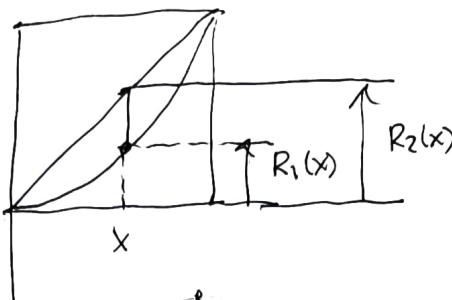
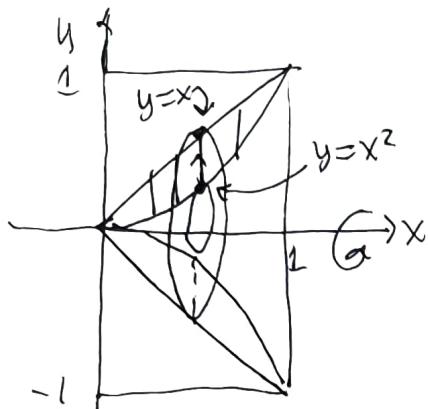
$$V = \int_0^1 \pi y^4 dy = \pi \frac{y^5}{5} \Big|_0^1 = \frac{\pi}{5}$$

6.2

Volumes (of revolution etc)

④

rotate region between 2 curves around either axis:  
 linear cross-section rotates into a "washer" (annulus!) instead of  
 a "disk" (circle!)



Identify inner and outer radii of annular cross-section

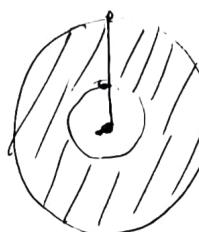
$$\begin{aligned} A(x) &= \pi R_2(x)^2 - \pi R_1(x)^2 \\ &= \pi(R_2(x)^2 - R_1(x)^2) \end{aligned}$$

$$R_1(x) = x^2$$

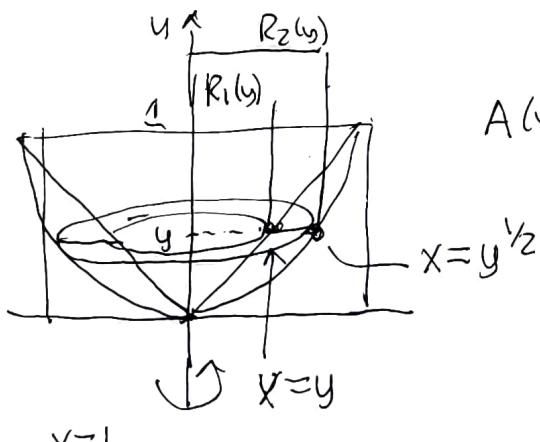
$$R_2(x) = x$$

$$\begin{aligned} A(x) &= \pi(x^2 - (x^2)^2) \\ &= \pi(x^2 - x^4) \end{aligned}$$

$$V = \int_0^1 \pi(x^2 - x^4) dx = \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right) \Big|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$$



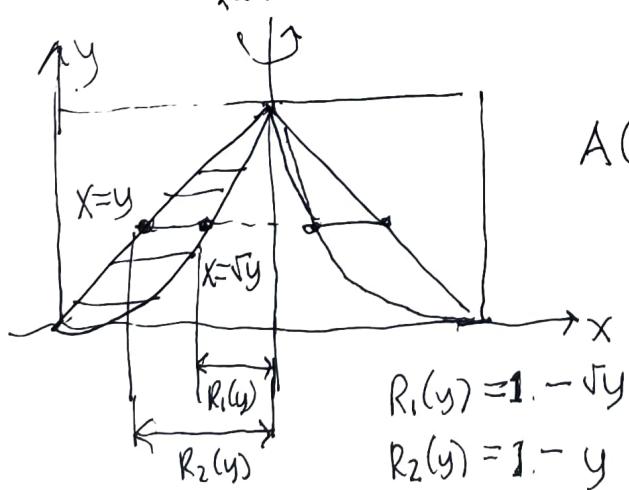
OR



$$\begin{aligned} A(y) &= \pi(R_2(y)^2 - R_1(y)^2) \\ &= \pi((y^{1/2})^2 - y^2) \\ &= \pi(y - y^2) \end{aligned}$$

$$V = \int_0^1 \pi(y - y^2) dy = \text{etc.}$$

OR

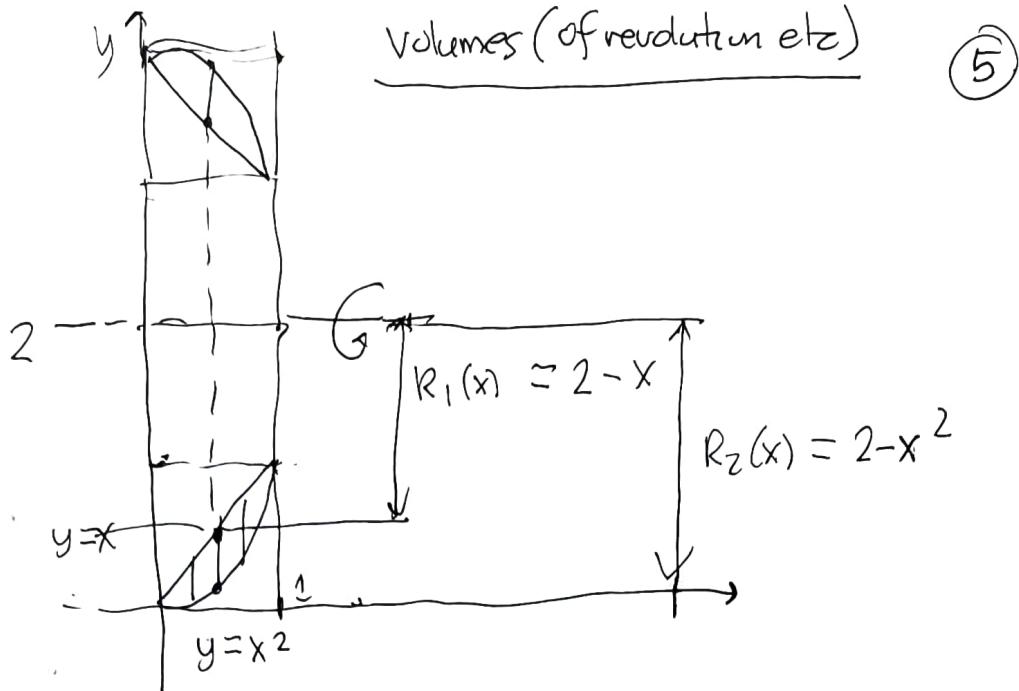


$$A(y)$$

$$\begin{aligned} &= \pi((1-y)^2 - (1-\sqrt{y})^2) \\ &= \pi(1-2y+y^2 - (1-2y^{1/2}+y)) \\ &= \pi(y^2 - 3y + 2y^{1/2}) \end{aligned}$$

$$V = \int_0^1 \pi(y^2 - 3y + 2y^{1/2}) dy = \dots$$

6.2 | 5



$$\begin{aligned}
 A(x) &= \pi ((2-x)^2 - (2-x^2)^2) \\
 &= \pi (4 - 4x^2 + x^4 - (4 - 4x + x^2)) \\
 &= \pi (4x - 5x^2 + x^4)
 \end{aligned}$$

$$V = \int_0^1 A(x) dx = \dots$$

Why? Helps you understand how to translate information about relationships between  $x$  &  $y$  and how to describe the rotated regions.

Practice in deciphering how to geometrically describe plane regions representing solids.

(5)