

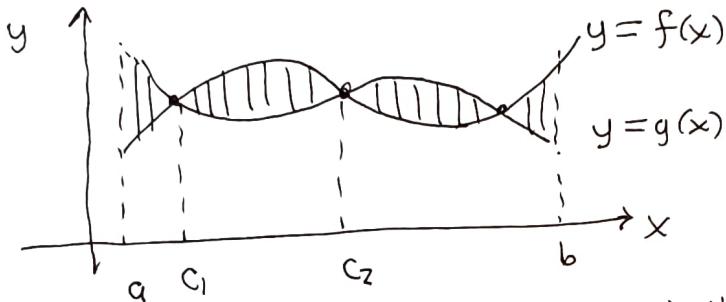
6.1

Areas between curves

1

The Riemann sum limit definition defines what we mean by the net area between a function graph and the horizontal axis over an interval while the Fundamental theorem of calculus gives us a concrete method to evaluate this.

We can use the Riemann sum approach to derive integral formulas for more complicated quantities that can be described by integration. The easiest such quantity to consider is the positive area of a region of the plane enclosed by two function graphs over a given interval.

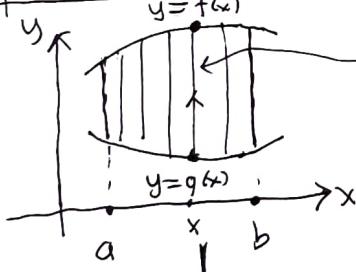


This requires finding the curve intersections within the interval, breaking the integral into a sum of integrals over the subintervals, and identifying the upper & lower function graph on each:

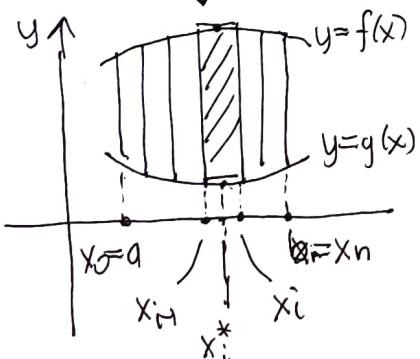
$$A = \int_a^b |f(x) - g(x)| dx = \int_a^b \underbrace{\text{upper}(x) - \text{lower}(x)}_{\substack{\text{piecewise defined} \\ \text{integrand}}} dx$$

↑ generates piecewise function

simplest case: $f(x) \geq g(x)$ for $a \leq x \leq b$



typical vertical linear cross-section of this region labeled by starting and stopping equations for the y-variable which moves upwards when increasing (arrow!) the "bullet points" are labeled by these equations



$$\Delta x = \frac{b-a}{n}$$

Riemann approach

$$\Delta A_i \approx (f(x_i^*) - g(x_i^*)) \Delta x$$

$$A \approx \sum_{i=1}^n \Delta A_i = \sum_{i=1}^n (f(x_i^*) - g(x_i^*)) \Delta x$$

$$\downarrow A = \lim_{n \rightarrow \infty} \sum_{i=1}^n (f(x_i^*) - g(x_i^*)) \Delta x$$

$$\equiv \int_a^b \underbrace{f(x) - g(x)}_{L(x) = \text{length of vertical cross-section}} dx \quad \leftarrow \text{goal!}$$

aside: $\boxed{= \int_a^b f(x) dx - \int_a^b g(x) dx = \text{difference of signed areas under graphs}}$

6.1] Areas between curves

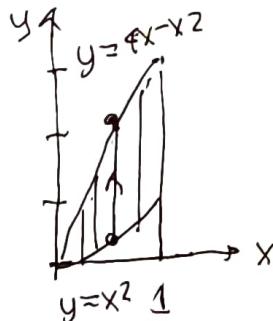
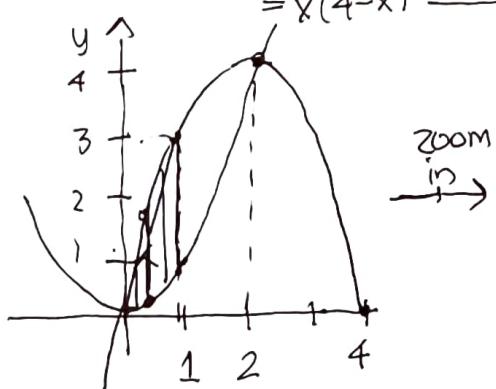
(2)

example $f(x) = 4x - x^2$, $g(x) = x^2$, $0 \leq x \leq 1$

$$= x(4-x) \xrightarrow{\text{intersections}} 4x - x^2 = x^2 \rightarrow 2x^2 - 4x = 0$$

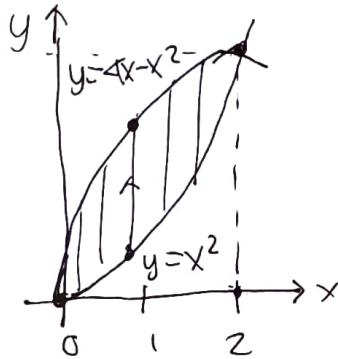
$$2x(x-2) = 0$$

$$x=0, x=2$$



$$A = \int_0^1 (4x - x^2) - x^2 dx = \int_0^1 4x - 2x^2 dx = \left[\frac{4x^2}{2} - \frac{2x^3}{3} \right]_0^1 = 2 - \frac{2}{3} = \frac{4}{3}$$

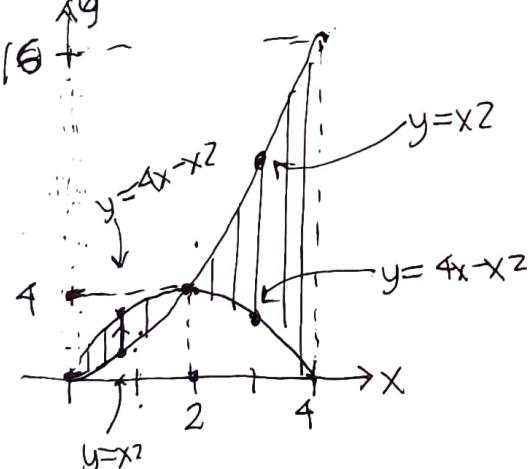
example same setup but now find area enclosed by function graphs
we already found the intersection points



$$A = \int_0^2 (4x - x^2) - x^2 dx = \int_0^2 4x - 2x^2 dx$$

$$= 2x^2 - \frac{2x^3}{3} \Big|_0^2 = 2 \cdot 2^2 - \frac{2}{3} 2^3 = 2^3 \left(1 - \frac{2}{3}\right) = \frac{2^3}{3} = \frac{8}{3}$$

example same setup but now find area between graphs for $0 \leq x \leq 4$
we found the intersection points



$$A = \int_0^2 (4x - x^2) - x^2 dx + \int_2^4 x^2 - (4x - x^2) dx$$

$$= \dots \quad (\text{who cares what the number is})$$

$$\begin{bmatrix} = & \frac{8}{3} + \frac{40}{3} = \frac{48}{3} = 16 \end{bmatrix}$$

our job is setting up the integrals!

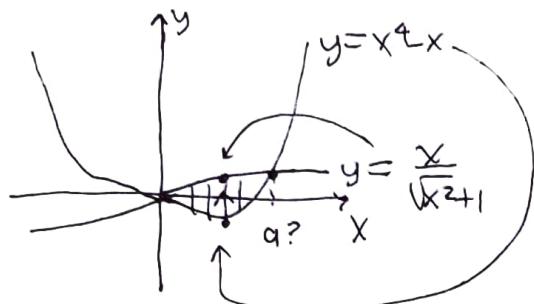
6.1

Areas between curves

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example (numerical root finding)

$$y = \frac{x}{\sqrt{x^2+1}}, y = x^4 - x = x(x^3 - 1)$$



graphically determine crossing situation
need to determine crossing point a
numerically

$$x^4 - x = \frac{x}{\sqrt{x^2+1}} \rightarrow x=0, a>0$$

↓ Maple

$$a \approx 1.180775703$$

$$\begin{aligned} A &= \int_0^a -(x^4 - x) + \frac{x}{\sqrt{x^2+1}} dx \quad \xrightarrow{\text{side calc}} \\ &= -\left(\frac{x^5}{5} - \frac{x^2}{2}\right) + \sqrt{1+x^2} \Big|_0^a \\ &= -\left(\frac{a^5}{5} - \frac{a^2}{2}\right) + \sqrt{1+a^2} - 1 \end{aligned}$$

$$\left\{ \begin{array}{l} \int \frac{(x dx)}{\sqrt{x^2+1}} du^{1/2} = \int u^{-1/2} \frac{du}{2} = \frac{1}{2} u^{1/2} \\ u = 1+x^2 \\ du = 2x dx \\ \frac{dx}{2} = x dx \end{array} \right.$$

$$= u^{1/2} = \sqrt{1+x^2}$$

$$= -1 - \frac{a^5}{5} + \frac{a^2}{2} + \sqrt{1+a^2} \quad \approx 0.785388550$$

Maple

$$\approx 0.785$$

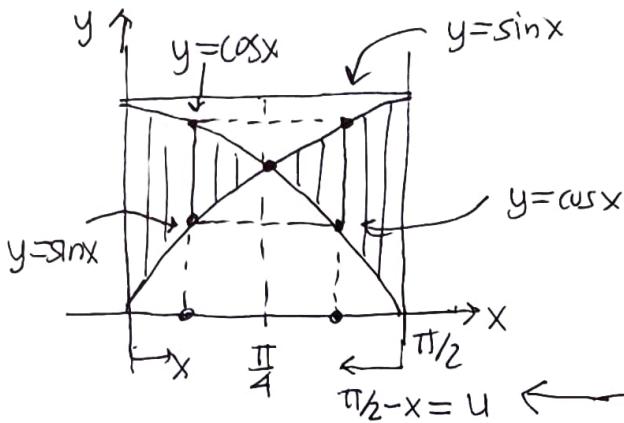
need specified #
digits to shorten!

6.1 Areas between Curves

(4)

Symmetry considerations

example Find area between $\cos x$ and $\sin x$ for $0 \leq x \leq \pi/2$.



We break up the interval into 2 equal subintervals,
clearly the areas reflect across
the center dividing line.

for second integral use new variable

$$A = \int_0^{\pi/4} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx$$

$x = \pi/4 \rightarrow u = \pi/4$
 $u = \frac{\pi}{2} - x \rightarrow x = \frac{\pi}{2} - u \quad x = \pi/2 \rightarrow u = 0$

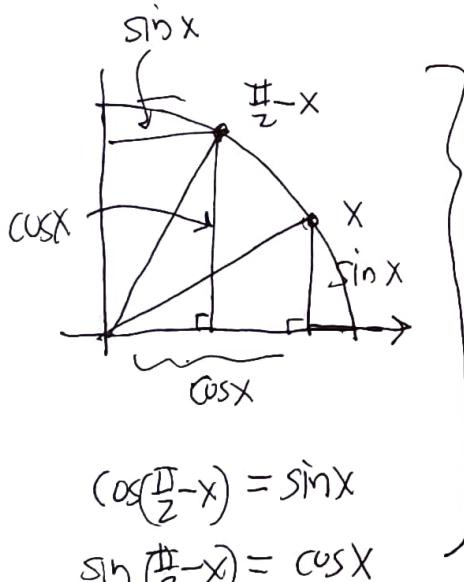
$$du = -dx$$

$$= \int_{\pi/4}^0 (\sin(\frac{\pi}{2}-u) - \cos(\frac{\pi}{2}-u))(-du)$$

$$= (-1)(-1) \int_0^{\pi/4} (\underbrace{\sin(\frac{\pi}{2}-u)}_{\cos u} - \underbrace{\cos(\frac{\pi}{2}-u)}_{\sin u}) du$$

$$= \int_0^{\pi/4} \cos u - \sin u \, du$$

$$= \int_0^{\pi/4} \cos x - \sin x \, dx$$



tng identities
complementary
angles

$$\begin{aligned} \text{so } A &= 2 \int_0^{\pi/4} \cos x - \sin x \, dx = 2(\sin x + \cos x) \Big|_0^{\pi/4} \\ &= 2(\sin \frac{\pi}{4} + \cos \frac{\pi}{4}) - 2(\sin 0 + \cos 0) = 2(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1) \\ &= 2\sqrt{2} - 2 \approx 0.828 \\ &\quad > 0! \end{aligned}$$

6.1 Areas between Curves

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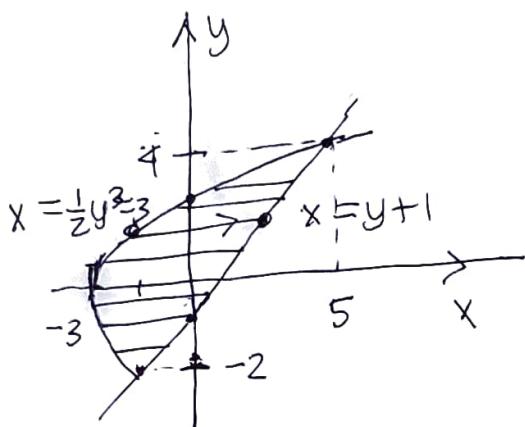
x & y in x - y plane can be re-interpreted!
variable names unimportant. concepts universal.

$$y = x - 1, \quad y^2 = 2x + 6 \rightarrow \text{area between curves?}$$

$$\downarrow \qquad \qquad \qquad \hookrightarrow x = \frac{y^2 - 6}{2}$$

$$x = y + 1 \qquad \qquad \qquad = \frac{1}{2}y^2 - 3$$

really y is independent variable here, x is dependent



$$\begin{aligned} \text{Intersection: } y + 1 &= \frac{1}{2}y^2 - 3 \\ 2y + 2 &= y^2 - 6 \\ y^2 - 2y - 8 &= 0 \\ y &= \frac{2 \pm \sqrt{2^2 - 4(-8)}}{2} \\ &= \frac{2 \pm \sqrt{36}}{2} = \frac{2 \pm 6}{2} = -2, 4 \end{aligned}$$

$$A = \int_{-2}^4 (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy$$

= ... etc.