

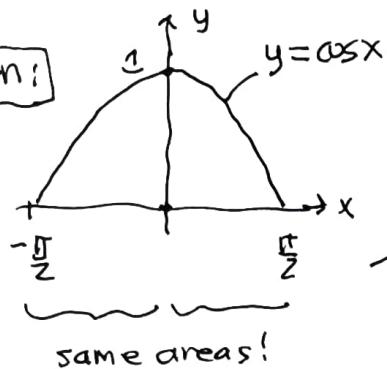
5.5b symmetry / transforming definite integrals

①

Symmetry and reflected intervals

Even and odd functions integrated over an interval centered at zero have special properties.

even:

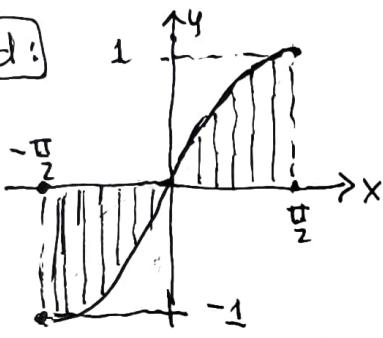


$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx = 2 \int_0^{\frac{\pi}{2}} \cos x dx$$

$$= 2 \sin x \Big|_0^{\frac{\pi}{2}} = 2 \sin \frac{\pi}{2} - 0 = 2$$

same areas!

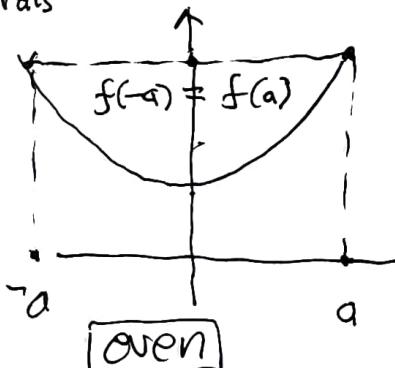
odd:



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x dx = 0 \quad \text{cancel!}$$

areas equal but
opposite signs as
integrals

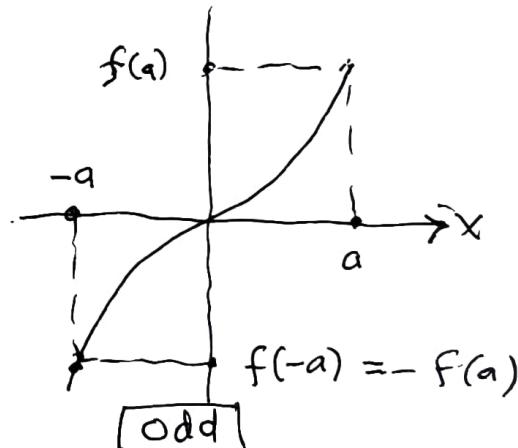
Why?



even

$$f(-x) = \pm f(x)$$

The same is true for even or odd powers.



odd

$$\int_{-a}^a f(x) dx = \underbrace{\int_{-a}^0 f(x) dx}_{\begin{cases} \text{let } x = -u, dx = -du \\ f(x) = f(u) = \pm f(u) \\ x = 0 \rightarrow u = 0 \\ x = -a \rightarrow u = a \end{cases}} + \int_0^a f(x) dx = \pm \int_0^a f(x) dx + \int_0^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx \\ 0 \end{cases}$$

$$\int_a^0 \mp f(u) du = \pm \int_0^a f(u) du$$

switch ↑ "dummy" variable

5.5b symmetry / transforming definite integrals

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Transforming integrals is important

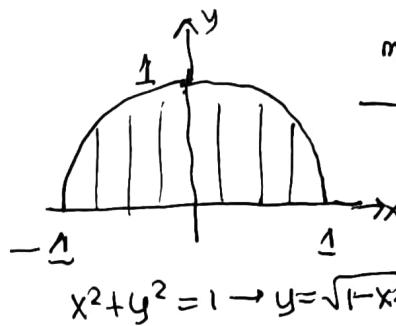
unit circle (upper)

"scale" length units by $a > 0$:

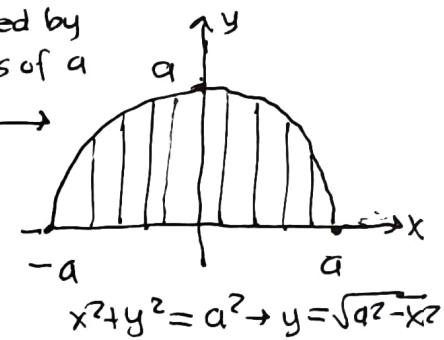
$$x \rightarrow ax$$

$$y \rightarrow ay$$

area scales by a^2



unit tickmarks
replaced by
multiples of a



$$\text{Area} = \int_{-a}^a \sqrt{a^2 - x^2} dx = \underset{\text{symmetry}}{2} \int_0^a \sqrt{a^2 - x^2} dx$$

$$= 2 \int_0^a \sqrt{a^2 \left(1 - \frac{x^2}{a^2}\right)} dx = 2a \int_0^a \sqrt{1 - \left(\frac{x}{a}\right)^2} dx$$

$$\left. \begin{array}{l} \text{let } u = \frac{x}{a} \text{ dimensionless variable} \\ \text{(length units cancel)} \\ du = \frac{dx}{a} \rightarrow dx = a du \\ x=0 \rightarrow u=0 \\ x=a \rightarrow u=1 \end{array} \right\}$$

$$= 2a \int_0^1 \sqrt{1-u^2} a du$$

$$= a^2 \left(\int_0^1 2\sqrt{1-u^2} du \right)$$

just a numerical coefficient — can be approximated if necessary

dependence on parameter
fixed to be a^2 times
some number

possible by introduction
of dimensionless variable

$$\text{Area} = a^2 \left(u \sqrt{1-u^2} + \arcsin u \right) \Big|_0^1 = \frac{\pi}{2} a^2$$

Maple
antiderivative

$$\begin{aligned} &\underbrace{1(0) + \arcsin 1 - 0(1) - \arcsin(0)} \\ &= \arcsin 1 = \pi/2 \end{aligned}$$

half area of circle

Often introducing dimensionless variables (ratios whose units cancel)
leads to a numerical integral times some function of the parameters
in applications.

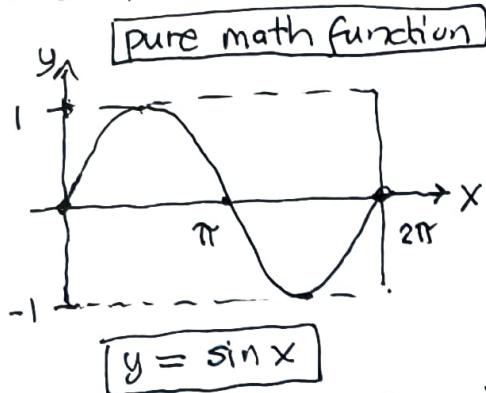
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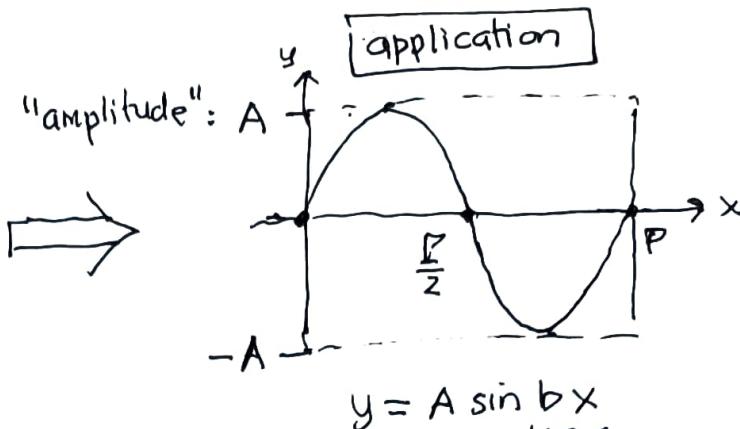
Applications always introduce positive "scaling" parameters to set the scale of the x and y axes in a functional relationship

$$y = f(x).$$

Example: oscillations



periodic: $\sin(x+2\pi) = \sin x$
period 2π



period: $bx = 2\pi \rightarrow x = \frac{2\pi}{b} \equiv P$

$$y = A \sin\left(\frac{2\pi x}{P}\right)$$

or

$$\frac{y}{A} = \sin\left(2\pi\left(\frac{x}{P}\right)\right)$$

$$\frac{y}{A} = \sin 2\pi \frac{x}{P}$$

ratios are
"dimensionless variables"
(dimensions cancel)

The amplitude and period are physically interpretable parameters.

unit tickmarks in y \rightarrow multiples of A
multiples of 2π in x \rightarrow multiples of P } but shape of graph unchanged!

What is the area under one half "cycle" (i.e., period) of the sine curve in the application?

$$\text{Area} = \int_0^{P/2} A \sin \frac{2\pi x}{P} dx \quad \begin{array}{l} \text{let } u = \frac{2\pi x}{P} \rightarrow du = \frac{2\pi}{P} dx \\ x=0 \rightarrow u=0 \\ x=P/2 \rightarrow u=\pi \end{array} \quad \begin{aligned} &= \int_0^{\pi} A \sin u \left(\frac{P}{2\pi} du\right) \\ &= AP \left(\frac{1}{2\pi} \int_0^{\pi} \sin u du \right) \\ &= -\frac{1}{2\pi} (\cos u) \Big|_0^{\pi} = \frac{1}{2\pi} (1 - (-1)) = \frac{1}{\pi} \end{aligned}$$

$$\text{Area} = \frac{AP}{\pi}$$

integral transformed to a numerical coefficient times function of parameters

just a number!
 ≈ 0.3183

5.5b

symmetry / transforming definite integrals

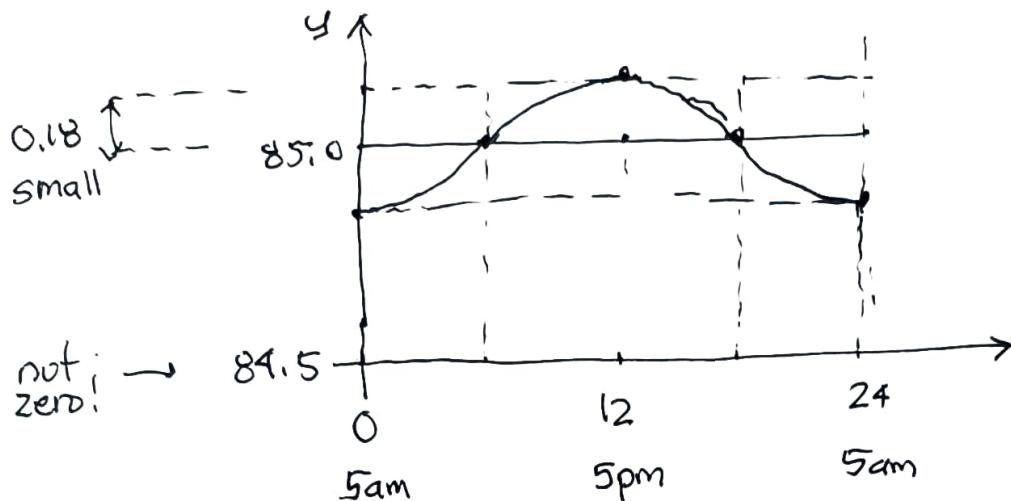
4

Example Stewart 8e: 5.5.80

The basal-metabolism rate (derivative) in units of kcal/h
energy time

depends on the time of day:

$$t = 0 \dots 24 = P \text{ (period)} \Leftrightarrow \text{time after 5am}$$



$$R(t) = 85 - 0.18 \cos \frac{\pi t}{12} = \frac{d B(t)}{dt}$$

amplitude

"total" basal-metabolism:

$$= \int_0^{24} R(t) dt = \text{net change over 24 hours}$$

= total energy expended by body

$$= \int_0^{24} 85 - 0.18 \cos \frac{\pi t}{12} dt$$

$$= 85t - 0.18 \left[\frac{\sin \frac{\pi t}{12}}{\pi/12} \right]_0^{24}$$

$$= 85(24-0) - \frac{12(0.18)}{\pi} (\underbrace{\sin 2\pi - \sin 0}_0)$$

$$\int \cos ax dx = \frac{1}{a} \sin ax + C$$

trivial u-sub!

$$= 85(24)$$

$$= 2016 \text{ kcal}$$

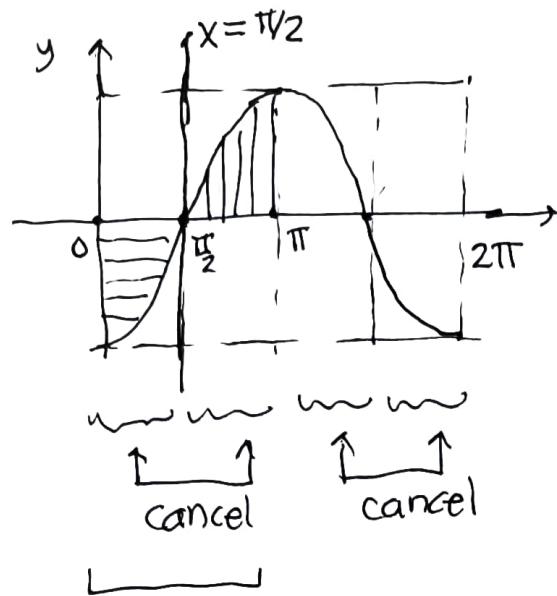
applications usually have
physical units

5.5b

symmetry / transforming definite integrals

(5)

Why did the integral of the oscillation term vanish?

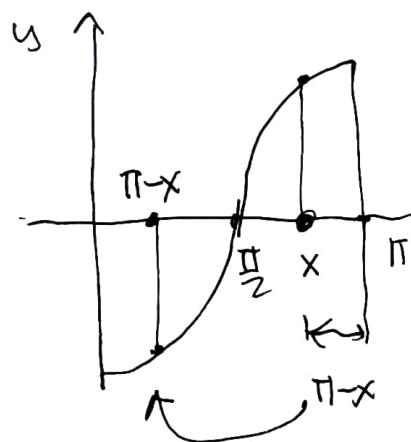


$$y = -\cos x \quad (\text{remove parameters})$$

$$\int_0^{2\pi} -\cos x \, dx = 0$$

$$-\cos(\pi - x) = +\cos x$$

↑ ↑
odd when reflected across
vertical line $x = \frac{\pi}{2}$:



$$x \rightarrow \pi - x$$

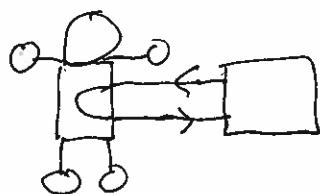
same principle as
odd under $x \rightarrow -x$
but translated
to the right.

5.5b) symmetry / transforming definite integrals

(6)

Exercise 5.5.85

Dialysis removes urea etc from blood cycling all body blood thru machine (dialyzer).



3 parameters

① $r \left(\frac{\text{mL}}{\text{min}} \right)$ = rate of blood flow thru machine

② $V (\text{mL})$ = total amount of blood

③ $C_0 (\text{mL})$ = total amount of urea etc (eventually all removed)

$$T = \frac{V}{r} \left(\frac{\text{mL}}{\text{mL/min}} = \text{min} \right) = \text{time it takes for all blood to cycle thru machine once}$$

Let $\mathcal{U}(t)$ = total amount of urea etc removed from blood

$\mathcal{U}(0) = 0$ initially

$\mathcal{U}(\infty) = C_0 \leftarrow$ all removed eventually

FACT: $\frac{d\mathcal{U}}{dt} = \frac{r C_0}{V} e^{-\frac{rt}{V}} = \frac{C_0}{T} e^{-\frac{t}{T}}$

goes from 1 to 0
 exponential decay

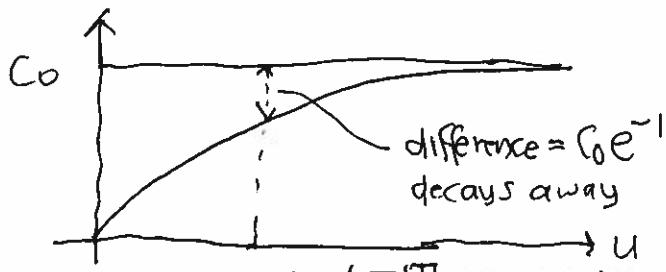
How much removed in 30 min?

$$\mathcal{U}(30) = \Delta \mathcal{U} = \int_0^{30} \frac{d\mathcal{U}}{dt} dt = \int_0^{30} \frac{C_0}{T} e^{-\frac{t}{T}} dt \quad \begin{matrix} \text{initial} \\ \text{rate at } t=0 \end{matrix} \quad \begin{matrix} \text{only } C_0, T \\ \text{enter integral} \end{matrix}$$

Let $u = \frac{t}{T}$, $t = uT$ = time measured in multiples of the cycle time
 (natural tickmarks for problem!)

$$du = \frac{dt}{T}, \quad t=0 \rightarrow u=0, \quad t=30 \rightarrow u=30/T$$

$$\begin{aligned} \mathcal{U}(30) &= \int_0^{30} C_0 e^{-\frac{t}{T}} dt = C_0 \int_0^{30/T} e^{-u} du = C_0 e^{-u} \Big|_0^{30/T} \\ &= C_0 (1 - e^{-30/T}) = C_0 (1 - e^{-u}) \Big|_{u=0}^{u=30/T} \end{aligned}$$



$0 \text{ as } u \rightarrow \infty$

$= 0$ initially

Final answer:
 $\mathcal{U}(30) = C_0 (1 - e^{-\frac{30}{T}})$

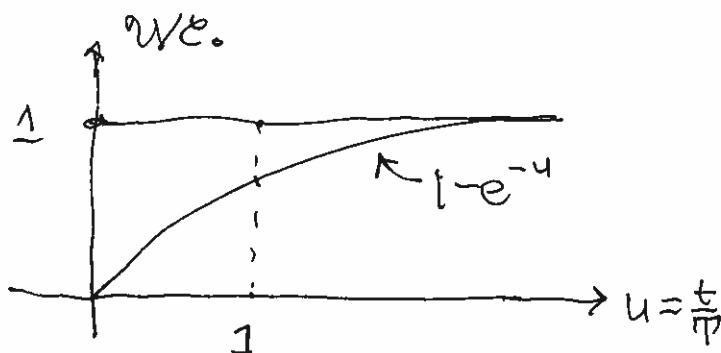
5.5b symmetry / transforming definite integrals (7)

For any time:

$$U(t) = C_0 \left(1 - e^{-\frac{t}{T}}\right)$$

$$\underbrace{\frac{U}{C_0}}_{\text{dimensionless variable}} = 1 - e^{-u} \quad \begin{matrix} u = \frac{t}{T} & \text{dimensionless time variable} \\ (\# \text{ cycles of machine}) \end{matrix}$$

(fraction of total area etc)



exponential rise
turn on function
from 0 to 1
(asymptotically)

In practice reaches 1%
of final amount when

$$1 - e^{-u} = 0.99$$

↪ solve $u \approx 4.6$ cycles

dimensional analysis / variables are often
fundamental in UNDERSTANDING a system!