5.4 Indefinite integrals and net change

we need a notation for antidifferentiation

$$f(x) \longrightarrow F(x) \quad \text{such that } F'(x) = f(x).$$

We define "the indefinite integral" of f wrt x:

"in" since no definite interval of integration is specified

$$\int f(x)\,dx = \underbrace{\underbrace{F(x)}_{\text{any particular antiderivative}} + C}_{\text{this is a 1-parameter family of antiderivatives where } C \text{ is the parameter describing this family}}$$

where C is an "arbitrary" constant

integral has 2 parts!
$\underbrace{\text{int}}_{①}$ wrt $\underbrace{x}_{②}$

by definition

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = \frac{d}{dx}(F(x)+C) = F'(x) + 0 = f(x)$$

so "d/dx" undoes $\int \ldots dx$

and
$$\int \frac{d}{dx}[f(x)]\,dx = f(x) + C$$

so "int wrtx" undoes "d/dx", apart from the additive constant

So these are "almost" inverse operations.

Notationally
$$\underbrace{\int_a^b f(x)\,dx}_{\text{definite integral}} = \underbrace{\left[\int f(x)\,dx\right]\Big|_a^b}_{\text{difference of values of indefinite integral}}$$

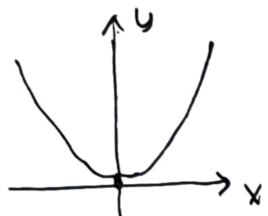
since the RHS equals:

$$[F(x) + C]\Big|_a^b = F(x)\Big|_a^b + 0$$
$$= F(x)\Big|_a^b$$

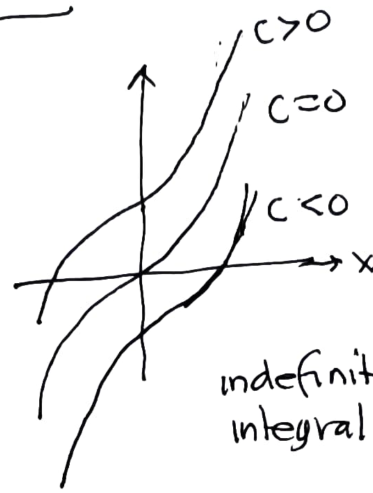
5.4 Indefinite integrals and net change

Example (power rule)

$$\int x^2\,dx = \underbrace{\frac{x^3}{3} + C}_{} \longrightarrow \frac{d}{dx}\left(\frac{x^3}{3}+c\right) = x^2 + 0 = x^2$$



original function



indefinite integral

family of functions whose graphs are translated vertically from any given member of the family

For a definite integral:

$$\int_1^2 x^2\,dx = \underbrace{\frac{x^3}{3}}_{}\Big|_1^2 = \frac{2^3-1^3}{3} = \frac{8-1}{3} = \frac{7}{3}$$

pick any antiderivative (i.e., ignore the arbitrary constant)

Conversely:

$$\int \frac{d}{dx}(x^2)\,dx = \int 2x\,dx = 2\left(\frac{x^2}{2}\right) + C = x^2 + C$$

5.4 indefinite integrals and net change

The **net change** of a quantity is the definite integral of its rate of change:

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ where } F'(x) = f(x) \text{ so:}$$

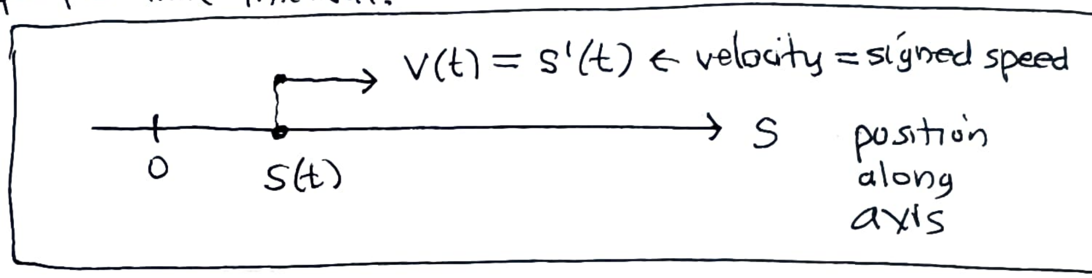
$$\int_a^b F'(x)\,dx = F(b) - F(a) \equiv \Delta F$$

symbol for change in quantity.

if we know the rate of change of a quantity, its integral gives the net change over an interval.

Example: we can read off the speed of a car from its speedometer, and the integral of speed is distance traveled over a specific time interval.

1-d motion



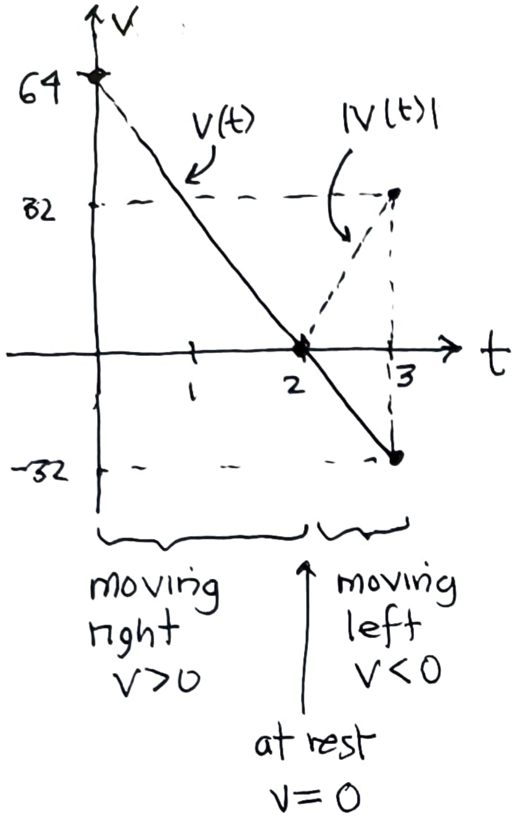
$$\int_a^b v(t)\,dt = \int_a^b s'(t)\,dt = s(t)\Big|_a^b = s(b) - s(a)$$ "displacement" (net change in position)

$$\int_a^b |v(t)|\,dt = \text{distance traveled}$$

abs. value signs convert to a piecewise function which complicates antidifferentiation
so we need to break up the interval to intervals over which the integrand has a definite sign

Indefinite integrals and net change

Example



$$v(t) = 64 - 32t, \quad 0 \le t \le 3$$

$$|v(t)| = \begin{cases} 64-32t, & 0 \le t \le 2 \\ -(64-32t), & 2 \le t \le 3 \end{cases}$$

Note:

$$s(t) = \int v(t)\,dt = \int 64-32t\,dt$$
$$= 64t - 16t^2 + C$$

$s(0) = C$ so

$$s(t) = 64t - 16t^2 + s(0)$$
$$s(t) - s(0) = \underbrace{64t - 16t^2}_{\text{antiderivative is net change in position from 0 to } t}$$

Net change:

$$s(3) - s(0) = \int_0^3 v(t)\,dt = \int_0^3 64-32t\,dt = 64t - 16t\Big|_0^3 = 48$$

distance traveled:

$$\int_0^3 |v(t)|\,dt = \int_0^2 v(t)\,dt - \int_2^3 v(t)\,dt = \int_0^2 64-32t\,dt - \int_2^3 64-32t\,dt$$
$$= \underbrace{(64t - 16t^2)\Big|_0^2}_{64} \underbrace{- (64t - 16t^2)\Big|_2^3}_{+16} = 80$$

Visualization: