$\boxed{11.R}$ Taylor series : the big picture

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a| < R$$

1) One can evaluate these coefficients one by one $n = 0, 1, 2, \ldots$ by direct differentiation BUT only if one can find a formula as a function of n for c_n does one get an explicit expression for the infinite series.
Without that formula, one can still use the explicit lowest terms of the series to approximate $f(x)$ in many applications.

2) "Tricks"
Once we have some power series representations, we can "manipulate" them to extend them to more complicated functions
Starting from the geo series sum $f(x) = (1+x)^{-1} = \sum_{n=0}^{\infty}(-1)^n x^n, \ |x|<1$
or its binomial series generalization

$$f(x) = (1+x)^{-k} = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)\,x^n}{n!}, \quad |x| < 1$$

we can combine various operations to generalize them to more complicated series

a) algebra : constants

$$f(x) = a(b+cx)^{-k} = a\left[b\left(1+\tfrac{c}{b}x\right)\right]^{-k} = ab^{-k}(1+\underbrace{c/b\, x}_{x \to c/b x \text{ in binomial}})^{-k}$$

b) algebra : composition with power functions

$$x \longrightarrow Ax^p, \ p > 0 \text{ integer}$$

c) integration or differentiation term by term:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \to f'(x) = \sum_{n=0}^{\infty} c_n \frac{d}{dx}(x^n)$$

$$\to \int f(x)\,dx = \sum_{n=0}^{\infty} c_n \int x^n dx$$

d) combining those in any order.

3) Taylor approximations by truncating the series after the first few terms is often enough to answer useful questions in STEM apps, and when alternating, it also gives an estimate for the truncation error.

4) When we have an explicit formula for the nth Taylor series coefficient, and the interval of convergence does not follow from manipulation of the geometric or binomial series result (for the radius of convergence), we use the absolute convergence ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$ to determine the open interval of convergence

but at the endpoints where this limit equals 1 and the test fails, we must compare to p-series and alternating series to see if the endpoints are included in the convergence interval.