11.9 powerseries tricks

Before we discuss how to represent functions by power series, we explore a wide family of function representations that directly follow from manipulating the summation formula for geometric series.

First: $\frac{a}{1-r} = \sum_{n=1}^{\infty} a r^{n-1} = \sum_{m=0}^{\infty} a r^m = \sum_{n=0}^{\infty} a r^n$

reorder by power $\begin{cases} m = n-1 \\ n = m+1 \\ n=1 \to m=0 \end{cases}$ dummy index → change back

we already used this as an example

$a=1,\ r=x:\quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$

But we can make other substitutions which still result in a powerseries $\sum_{n=0} c_n x^n$.

1) Substitution algebra tricks

set $a = Ax^q$, $q \geq 0$ integer

$r = Bx^p$, $p \geq 1$ integer

$\hookrightarrow |Bx^p| < 1$ convergence

$\hookrightarrow |x| < |B|^{-1/p} = R$

$$\frac{a}{1-r} = \sum_{n=0}^{\infty} A x^q (Bx^p)^n = \sum_{n=0}^{\infty} A B^n x^{\underbrace{pn+q}_{m = np+q \text{ reorder}}}$$

$$\frac{a}{1-r} = \frac{Ax^q}{\underbrace{1-}_{\text{not necessary}}Bx^p}$$

If start with $\frac{Ax^q}{c+bx^p} = \frac{Ax^q}{c(1+\frac{b}{c}x^p)} = \frac{Ax^q}{c(1-\underbrace{(-\frac{b}{c})}_{B}x^p)}$ finish with upper formula.

2) We can include differentiation / integration term by term with the above algebra steps before, after, or during!

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(c_n x^n) = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$\int \left(\sum_{n=0}^{\infty} c_n x^n\right) dx = \sum_{n=0}^{\infty} \int (c_n x^n dx) \cdot \quad \sum_{n=0}^{\infty} c_n \frac{x^{n+1}}{n+1}$$

factors dont change abs. conv ratio test
same radius of convergence but endpoint convergence can change

factors of n, $n+1$ change limiting p-series factor comparison power series tricks

Examples show how all this is carried out.

example

$$\frac{1}{1+x^2} = \frac{1}{1-(\underbrace{-x^2}_{r})} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$ even series

even function

if $|-x^2|<1 \to |x|<1$

diverges when $x = \pm 1$: $\frac{1}{1+1} \neq \sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+\cdots$

integration

Next recall:

$$\int \frac{dx}{1+x^2} = \arctan x + C$$

[or $\arctan x = \int_0^x \frac{dt}{1+t^2}$]

$$= \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\,dx = \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^{2n+1}}{2n+1} + C_n\right)$$

$$= \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}_{=0 \text{ at } x=0 \text{ so}} + C_2 = \arctan x + C$$

but $\arctan(0) = 0$

$C_2 = C$

$= \arctan x$, $|x|<1$

(odd function = odd series)

endpoints: $x^2=1$: $x^{2n+1} = x^{2n}x = x$

$$\sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n(\pm 1)}{2n+1}}_{} = \arctan(\pm 1) = \frac{\pi}{2}, -\frac{\pi}{2}$$

$\underset{n\to\infty}{\sim} \pm\frac{(-1)^n}{2n}$ alternating harmonic series converges at endpoints

(integration can improve convergence at endpoints!)

differentiation

(next page)

$\boxed{11.9}$ power series tricks

$$\frac{d}{dx}\left[\underbrace{\frac{1}{1+x^2}}_{(1+x^2)^{-1}} = \sum_{n=0}^{\infty}(-1)^n x^{2n}\right], \quad |x|<1$$

$$-\frac{1}{2}\left[\underbrace{-(1+x^2)^{-2}(2x)}_{\frac{-2x}{(1+x^2)^2}} = \sum_{n=0}^{\infty}(-1)^n(2n)x^{2n-1}\right]$$

$n=0 \to n=1$

$$\frac{x}{(1+x^2)^2} = \sum_{n=1}^{\infty}\underbrace{\left(-\frac{1}{2}\right)(-1)^n(2n)}_{(-1)^{n+1}n}x^{2n-1}$$

$2n-1 \xrightarrow{n=1} 1$ starting value

$2m+1 \leftarrow m=0$

odd function.

$2n-1 = 2m+1$
$2n = 2m+2$
$n = m+1$
$n+1 = m+2$ ↑ useless

$$= \sum_{m=0}^{\infty}(-1)^{m+2}(m+1)x^{2m+1}$$

$$= \sum_{m=0}^{\infty}(-1)^m(m+1)x^{2m+1}$$

odd series. $|x|<1$

$|x|=1$: $\sum_{m=0}(-1)^m(m+1)(\pm 1)$ divergent!
$x=\pm 1$

worse than divergent series for original function $\pm(1-1+1-1\ldots)$

$\frac{1}{1-x}\ (x\to -x)$

$\boxed{\text{example}}$ $\int\left[\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n\right]dx$

$$\int\frac{1}{1+x}dx = \sum_{n=0}^{\infty}\int(-1)^n x^n dx$$

$$\ln(x+1)+C_1 = \left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}\right)+C_2 = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{x^m}{m}+C_2$$

$0 = \ln(0)+1+C_1 = 0+C_2$
$C_1 = C_2$

$m=n+1$
$n=m-1$
$n=0\to m=1$

$$\ln(1+x) = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{x^n}{n}$$

$\boxed{11.9}$ power series tricks

$\boxed{\text{example } 11.9.13}$

Goal: powerseries for $\dfrac{x^2}{(7+x)^3}$ from $\dfrac{1}{1-x}$

How to proceed step by step:

$$\frac{1}{7+x} \xrightarrow[\frac{d^2}{dx^2}]{(1)} \frac{1}{(7+x)^3} \rightarrow x^2\left(\frac{1}{(7+x)^3}\right)$$

↓ (divide out constants)

$$\frac{1}{7+x} = \frac{1}{7(1+\frac{x}{7})} = \frac{1}{7(1-(-\frac{x}{7}))} = \frac{1}{7}\sum_{n=0}^{\infty}\left(-\frac{x}{7}\right)^n$$

(algebra prestep)

$$\frac{1}{7+x} = (7+x)^{-1}$$

(1) $\downarrow \frac{d}{dx}$

$$-(7+x)^{-2} = \frac{1}{7}\sum\left(-\frac{1}{7}\right)^n \frac{d}{dx}x^n$$

$$\frac{1}{(7+x)^2} = \sum_{n=1}^{\infty}\left(-\frac{1}{7}\right)^{n+1} n x^{n-1}$$

$\downarrow \frac{d}{dx}$

$$-2(7+x)^{-3} = \sum_{n=2}^{\infty}\left(-\frac{1}{7}\right)^{n+1} n(n-1)x^{n-2}$$

$$\frac{1}{(7+x)^3} = \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)\left(-\frac{1}{7}\right)^{n+1} n(n-1)x^{n-2}$$

(2) $\downarrow \times x^2$

$$\frac{x^2}{(7+x)^3} = \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)\left(-\frac{1}{7}\right)^{n+1} n(n-1)x^{n-2+2}$$

$$= \sum_{n=2}^{\infty}(-1)^n \frac{n(n-1)}{2\cdot 7^{n+1}} x^n$$