$\boxed{11.9:}$ tricks with geometric series power series

starting point: $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$ when $|r|<1$

set $a=1, r=x$: $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$ $|x|<1$ or $-1<x<1$

$1+x+x^2+x^3+\dots = f(x)$

defines a function for $|x|<1$.

The following examples will show how we can generalize this to functions of the form $f(x) = \dfrac{Ax^p}{B+Cx^q}$, $p,q \geq 0$ integers.

EX $\dfrac{1}{2+x} = \dfrac{1}{2(1+\frac{x}{2})} = \dfrac{1}{2}\cdot\dfrac{1}{1-\underbrace{(-\frac{x}{2})}_{r}} = \dfrac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^n}{2\cdot 2^n}$

$= \sum_{n=0}^{\infty}\dfrac{(-1)^n x^n}{2^{n+1}}$ ✓ $\left|\frac{x}{2}\right|<1 \to |x|<2=R$ converges for $-2<x<2$

EX $\dfrac{1}{2+x^2} = \dfrac{1}{2(1+\frac{x^2}{2})} = \dfrac{1}{2}\cdot\dfrac{1}{1-\underbrace{(-\frac{x^2}{2})}_{r}} = \dfrac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x^2}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{2\cdot 2^n}$

$= \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{2^{n+1}}$ $\left|-\frac{x^2}{2}\right|<1$, $x^2<2$, $|x|<2^{1/2}=R$

so $-\sqrt{2}<x<\sqrt{2}$

EX $\dfrac{x^3}{2+x^2} = x^3\left(\dfrac{1}{2+x^2}\right) \overset{\downarrow}{=} x^3\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{2^{n+1}} = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+3}}{2^{n+1}}$

We can also combine these algebra manipulations with differentiation or integration of the resulting series.

Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$; $|x-a| < R$ defines f, where converges.

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

now differentiate:

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = \sum_{n=0}^{\infty} \underbrace{\frac{d}{dx}\left(c_n (x-a)^n\right)}_{c_n \cdot n (x-a)^{n+1}}$$

limit limit

interchange turns out to be valid

$$= \sum_{n=0}^{\infty} n\, c_n (x-a)^{n+1}$$

$$= \frac{d}{dx}\left(c_0 + c_1(x-a) + c_2(x-a)^2 + \dots\right)$$

$$= 0 + c_1(1) + c_2 \cdot 2(x-a)(1) + \dots$$

"term by term" differentiation

also converges on same interval (modulo endpts)

Similarly

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n\,dx = \sum_{n=0}^{\infty} \underbrace{\int c_n (x-a)^n\,dx}$$

limit limit

exchange

$$= \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

"term by term" integration

$$= \int \left(c_0 + c_1(x-a) + c_2(x-a)^2 + \dots\right) dx$$

$$= \underbrace{c_0 x + c_1 \frac{(x-a)^2}{2} + c_3 \frac{(x-a)^3}{3} + \dots}_{\downarrow} + C$$

also converges in same interval (modulo endpts)

and

$$\int_a^b f(x)\,dx = \left[\dots\right]\Big|_a^b$$

[11.9] tricks with geometric series power series

see text for derivative examples

EX $\quad \frac{1}{1+x} = \frac{1}{1-\underbrace{(-x)}_{r=-x}} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$ $\quad |x| < 1$

$-1 < x < 1$

$\xrightarrow{+1}$ $0 < \underbrace{x+1}_{\text{positive}} < 2$

$$\int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^n \, dx$$

$$= \sum_{n=0}^{\infty}(-1)^n \underbrace{\int x^n dx}_{\frac{x^{n+1}}{n+1}} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + C$$

$\ln(1+x)$ partiular antiderivative

choose C so RHS equals LHS

$x=0$: $\ln(1+0) = 0 + C$

$0 = C$

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$$

set $m = n+1 \rightarrow n = m-1$

$n=0 \rightarrow m=1$

$$= \sum_{m=1}^{\infty}(-1)^{m-1}\frac{x^m}{m} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

index symbol dues not matter

gives series for ln evaluation!

$0 < \underbrace{x+1} < 2$

values between 0 and 2.

$\boxed{11.9}$ tricks with geometric series power series

EX $\quad \dfrac{1}{1+x^2} = \dfrac{1}{\underbrace{1-(-x^2)}_{r=-x^2}} = \displaystyle\sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

$$\int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx$$

$$\downarrow \qquad = \sum_{n=0}^{\infty}\left(\int (-1)^n x^{2n}\,dx\right)$$

arctan x

particular antiderivative $\quad = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C$, $\quad$ for $|r| = |-x^2| < 1$

$x^2 < 1$

$|x| < \underline{1}$

0

set $x = 0$:

$\arctan 0 = 0 + C = C$

$$\underbrace{\arctan x}_{\text{odd function}} = \sum_{n=0}^{\infty}(-1)^n \underbrace{\frac{x^{2n+1}}{2n+1}}_{\text{only odd powers}}$$

11.9 tricks with geometric series power series

Definite integrals approximated with a power series.

$$\int_0^{0.3} \underbrace{\frac{x^5}{1+x^7}}_{\text{expand in series, then integrate}} dx$$

(Maple cannot find antiderivative!)

$$\frac{x^5}{1+x^7} = x^5 \cdot \frac{1}{1-(-x^7)} = x^5 \sum_{n=0}^{\infty} (-x^7)^n = \sum_{n=0}^{\infty} (-1)^n x^{7n+5}$$ } Geometric series trick

$$= x^5 - x^{12} + x^{19} - x^{26} + \cdots$$

$$\int \frac{x^5}{1+x^7}\,dx = \frac{x^6}{6} - \frac{x^{13}}{13} + \frac{x^{20}}{20} - \frac{x^{27}}{27} + \cdots + C$$ } term by term integration

$$\int_0^{0.3} \frac{x^5}{1+x^7}\,dx = [\;''\;]\Big|_{\underbrace{0}_{\text{lower limit contribution is zero}}}^{0.3} = \underbrace{\frac{.3^6}{6} - \frac{.3^{13}}{13} + \frac{.3^{20}}{20} - \frac{.3^{27}}{27} + \cdots}_{\text{alternating series}}$$ } definite integral evaluation

use Maple $= 0.\underbrace{000\,121}_{\text{6 dec places}}50 - \underbrace{1.2\cdot10^{-8} + 1.7\cdot10^{-12} - 2.8\cdot10^{-16}}_{}$

total error in rest of series $< 1.2\cdot10^{-8}$, shouldn't matter

but

0.00012148

≈ 0.000121 rounds down in 6th digit. (rare that this will happen)

→ evaluate these for last problem in WebAssign using Maple worksheet.