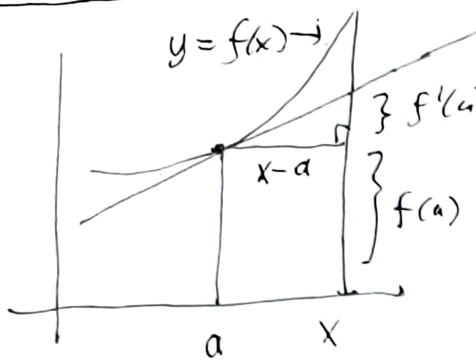
11.8 **Power series**

Motivation: linear approximation to a function



$y = f(a) + f'(a)(x-a)$ linear approximation

best for $|x-a|$ small enough.

How to improve approximation?
Go to higher order polynomials.

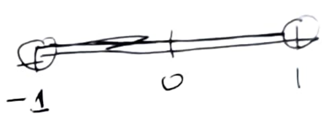
$$f(x) \approx \underbrace{f(a) + f'(a)(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_n(x-a)^n}$$

nth order polynomial, higher order → more coefficients to fit the function better.

$$= f(a) + f'(a)(x-a) + \dots$$ infinite series representation

example $\frac{1}{1-x} = \underbrace{\sum_{n=1}^{\infty} x^{n-1}}_{\substack{G.S \\ r=x \\ a=1}} = \underbrace{1+x}_{\text{linear approx.}} + x^2 + x^3 + \dots$ quadratic, cubic, etc. $|x|<1$

diverges elsewhere



interval of convergence.

Before we can connect up these series to existing functions, we need to study them and their convergence properties.

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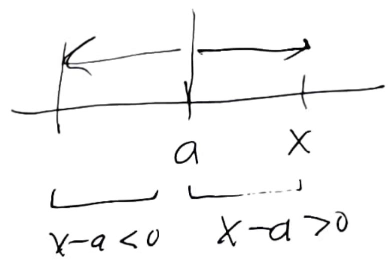
"powerseries centered at a"
or "about a"

$$\sum_{n=0}^{\infty} C_n(x-a)^n = C_0 \underbrace{(x-a)^0}_{\equiv 1} + C_1(x-a)^1 + C_2(x-a)^2 + \dots$$

coefficient

powers of $x-a$
"Geo series factor"



$x-a =$ signed distance from a.

$$\equiv \lim_{N\to\infty} \underbrace{\sum_{n=0}^{N} C_n(x-a)^n}$$

Nth order polynomial
= Nth partial sum

Use absolute convergence ratio test

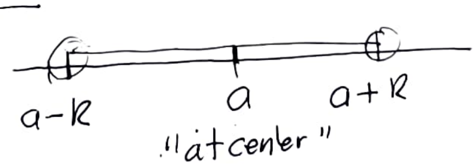
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|C_{n+1}|\,|x-a|^{n+1}}{|C_n|\,|x-a|^n} = \left|\frac{C_{n+1}}{C_n}\right| |x-a| \xrightarrow[n\to\infty]{\text{limit}} < 1$$

for convergence

places condition on $|x-a|$.

This process defines an "interval of convergence"

$\underbrace{|x-a| < R}$ ← "radius of convergence"



$a-R < x < a+R$

only 3 possibilities:

1) $R=0$, only converges at $x=a \rightarrow (x-a)=0$, only first term nonzero
2) $R=\infty \longrightarrow -\infty < x < \infty$ converges everywhere
3) $R>0$ finite: converges for $|x-a|<R$, diverges for $|x-a|>R$, ratio test fails, endpoints can do either, must analyze with other tests, like p-series

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example

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = \underbrace{\underbrace{\frac{(x-3)}{1}}_{\text{linear approx}} + \frac{(x-3)^2}{2}}_{\text{quad approx}} + \cdots \quad \text{etc}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{|x-3|^{n+1}}{n+1}}{\dfrac{|x-3|^n}{n}} = \frac{|x-3|^{n+1}}{n+1}\,\frac{n}{|x-3|^n} = \frac{n}{n+1}\,\frac{|x-3|^{n+1}}{|x-3|^n}$$

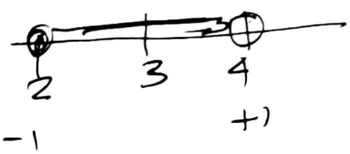
$$= \frac{1}{1+\frac{1}{n}}\,|x-3|^1 \longrightarrow |x-3| < 1 \text{ convergence}$$

($\frac{1}{n} \to 0$)

$|x-3| = 1$?

$x-3 = \pm 1$ plug back in

$$\sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n} \begin{cases} + & \displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \text{ harm div} \\ - & \displaystyle\sum_{n=1} \frac{(-1)^n}{n} \text{ alt harm. conv} \end{cases}$$



video example

Bessel function $J_0(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ centered at 0

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{2(n+1) \to 2n+2}}{\underbrace{2^{2(n+1)}}_{\underbrace{2^{2n+2}}_{2^{2n}\cdot 2^2}}(\underbrace{(n+1)!}_{(n+1)n!})^2} \cdot \frac{2^{2n}(n!)^2}{|x|^{2n}} = \frac{|x|^2}{2^2}\,\frac{\cancel{2^{2n}}\,\cancel{(n!)^2}}{\cancel{2^{2n}}(n+1)^2\cancel{(n!)^2}}$$

$$= \left|\frac{x}{2}\right|^2 \frac{1}{(n+1)^2} \longrightarrow 0$$

converges no matter what value x has, i.e., everywhere,
infinite radius of convergence

(see Maple worksheet)

example $\displaystyle\sum_{n=0}^{\infty} n!\,x^n \to \left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!\,|x|^{n+1}}{n!\,|x|^n} = \frac{(n+1)\cancel{n!}}{\cancel{n!}}\,\frac{\cancel{|x|^n}\cdot|x|}{\cancel{|x|^n}} = (n+1)\,|x|$

$\to \infty$ unless $|x| = 0$

diverges everywhere except at center

no matter how small the power of x (G.S!), the factorial overcomes it!

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example $\sum_{n=0}^{\infty} \frac{x^n}{n!} \to \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{\cancel{n!}}{(n+1)\cancel{n!}} |x|^1$

$= \frac{|x|}{n+1} \longrightarrow 0$

factorials beat G.S factors

G.S factors

example $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{\sqrt{n+1}}$

p-series factor

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}|x|^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{3^n |x|^n} = 3|x| \underbrace{\sqrt{\frac{n+1}{n+2}}}_{\to 1} \longrightarrow 3|x| < 1$

$\hookrightarrow |x| < \frac{1}{3}$

as expected.

endpoints! $|3x| = 1$

$3x = \pm 1 \longrightarrow x = 1/3$, $x = -1/3$

$\underbrace{\sum_{n=0}^{\infty} \frac{(\mp 1)^n}{\sqrt{n+1}}}$

$\underset{n\to\infty}{\sim} \frac{(\mp 1)^n}{n^{1/2}}$ — alt. $p = 1/2$ series conv

$p = 1/2$ series div.