11.6 the ratio test

Recall: geometric series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ar^{n-1} \xrightarrow{\text{abs value}}$ series $\sum_{n=1}^{\infty} |a||r|^{n-1}$

ratio $|r| = \left|\frac{a_{n+1}}{a_n}\right|$ fixed for all n

< 1 converges $\rightarrow$ converges absolutely

$= 1$ diverges

> 1 diverges $\quad |a_n| \not\rightarrow 0$! grows

Testing for absolute convergence is the tool we need to understand Taylor series representations of functions, and geometric series are the natural comparison series.

Ratio test for absolute convergence

$\sum_{n=1}^{\infty} a_n \longrightarrow$ suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ exists (or is ∞)

For large enough n, this looks very much like a geometric series with ratio $|r| = L$, and should have the same convergence properties by comparison.

$L < 1 \rightarrow$ it looks like a convergent geometric series and converges absolutely, so converges

$L > 1 \rightarrow$ it looks like a divergent geometric series. and diverges (successive terms are increasing!)

$L = 1 \rightarrow$ inconclusive

$L = \infty$ worse than any divergent geometric series so of course diverges

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Note: any p-series has limiting ratio of 1!

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad \lim_{n\to\infty} \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p$$

$$= \lim_{n\to\infty} \left(\frac{n}{(n+1)} \frac{/n}{/n}\right)^p = \lim_{n\to\infty} \left(\frac{1}{1+\underbrace{\tfrac{1}{n}}_{\to 0}}\right) = 1$$

yet converges for $p > 1$
diverges for $0 < p \leq 1$
so without extra analysis, the limiting ratio cannot determine if the series converges or diverges.

This is why the ratio test fails for $L = 1$.

example

$$\sum (-1)^{n+1} \frac{n^3}{3^n}$$

n^3 ← like a p-series factor (polynomial!)

3^n ← like a geometric series factor

$$3^n = (e^{\ln 3})^n = e^{(\ln 3) n}$$

(exponential!)

exponentials beat polynomials (l'Hôpital!)
so $|a_n| \to 0$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \frac{(n+1)^3}{n^3} \cdot \left(\frac{3^n}{3^{n+1}}\right) = \left(\frac{n+1}{n}\right)^3 \frac{3^n}{3^n \cdot 3}$$

$$= \left(1+\frac{1}{n}\right)^3 \frac{1}{3} \xrightarrow{n\to\infty} \frac{1}{3} < 1$$

$\underbrace{\frac{1}{n} \to 0}$ does not contribute to limit (always 1!)

looks like convergent GS, so converges!

$\boxed{11.6}$ the ratio test

example:

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

n^n ← both base and exponent increasing with n

$n!$ ← factorials increase fast but...

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} = \frac{\cancel{(n+1)}(n+1)^n}{n^n} \cdot \frac{\cancel{n!}}{\cancel{(n+1)}\,\cancel{n!}}$$

↑ expand ↑

$$= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \longrightarrow e^1 = e > 1$$

terms eventually increasing!

$\boxed{\text{diverges}}$

Recall:

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty} \left(e^{\ln(1+\frac{x}{n})}\right)^n$$

$$= \lim_{n\to\infty} e^{\ln(1+\frac{x}{n})\cdot n} = e^{\lim_{n\to\infty}\left(\frac{\ln(1+\frac{x}{n})}{\frac{1}{n}}\right)} = e^x$$

$\sim \frac{0}{0}$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{x}{n}} \cdot x\left(-\frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{x}{1+\frac{x}{n}} = x$$

$\frac{x}{n} \to 0$

$\boxed{11.6}$ the ratio test

the $\boxed{\text{root test}}$ is less important than the ratio test but is easier to evaluate the limiting ratio in cases involving factors with n in the exponent

geometric series:

$$\sum_{n=1}^{\infty} a r^{n-1} \to |a_n|^{\frac{1}{n}} = (|a||r|^{n-1})^{\frac{1}{n}}$$

$$= |a|^{\frac{1}{n}} |r|^{\frac{n-1}{n}}$$

$$= |a|^{\underbrace{\tfrac{1}{n}}_{0}} |r|^{1-\underbrace{\tfrac{1}{n}}_{0}} \xrightarrow{n\to\infty} = |a|^0 |r| = |r|$$

so $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |r|$ is another way of computing the limiting ratio for general series

$\boxed{\text{root test}}$:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \begin{cases} < 1 & \text{converges} \\ = 1 & \text{inconclusive} \\ > 1 & \text{diverges} \\ \infty & \text{diverges} \end{cases}$$

example

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n} \to |a_n|^{\frac{1}{n}} = \left(\frac{2^n}{n^n}\right)^{\frac{1}{n}} = \frac{2}{n} \xrightarrow{n\to\infty} 0$$

converges faster than any geometric series no matter how small the ratio.

11.6 the ratio test

But we will be using only the ratio test for taylor series so let's redo this using the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{2^{n+1}}{(n+1)^{n+1}}}{\dfrac{2^n}{n^n}} = \underbrace{\frac{2^{n+1}}{2^n}}\cdot\underbrace{\frac{n^n}{(n+1)^{n+1}}}$$

$$= 2 \qquad \frac{n^n}{(n+1)(n+1)^n} = \underbrace{\frac{1}{n+1}}\underbrace{\left(\frac{n}{n+1}\right)^n}$$

$$\frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e}$$

$$= 2 \cdot \underbrace{\frac{1}{(n+1)\,e}}$$

from n^n factor in denominator

from G.S. factor in numerator 2^n (obvious!)

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so we get a hierarchy of factors that contribute to the ratio: $\left|\frac{a_{n+1}}{a_n}\right|$

$$|a_n| = \frac{f_1 \cdots}{f_2 \cdots}$$

$\left|\frac{a_{n+1}}{a_n}\right|$:

n^p (or any polynomial since largest power dominates) $\xrightarrow[n\to\infty]{}$ 1

r^n (G.S. factor) $\xrightarrow[n\to\infty]{}$ $|r|$

$n!$ $\xrightarrow[n\to\infty]{}$ $\frac{(n+1)!}{n!} = n+1 \to$ grows

n^n $\xrightarrow[n\to\infty]{}$ $\underbrace{e}\cdot(n+1) \to$ grows faster

extra G.S. contribution to ratio.

lower factor types grow faster than higher types